Fall 2009 8b Midterm 1 Review Solutions

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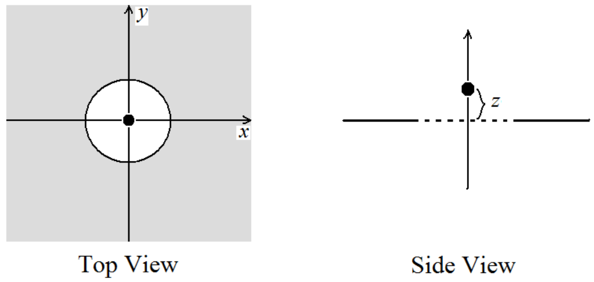
Contents

1 Charge Oscillator: Plane with a Hole



In the x-y plane (at $z = 0$) lies an infinite plane of constant surface charge density $\sigma > 0$ with a hole of radius R centered at the origin cut out of it. At the center of this hole, we place a negative point charge $-q$ of mass m. We displace the point charge vertically a very small amount z ($z > 0$ means we pull the charge up and $z < 0$ means we push it down), and then we let go. Your task is to describe the subsequent motion of the charge. To do this, do the following

(a) First, suppose that there is no hole in the infinite plane. Recall or rederive using Gauss' Law the electric field produced at the position of the point charge at $(0, 0, z)$. Don't forget to write both the magnitude and direction.

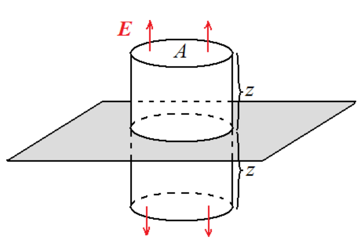
(b) Now, calculate the electric field produced at the same point $(0, 0, z)$ by a uniformly charged disc of charge density $-\sigma$, radius R and placed in the x-y plane centered at the origin.

(c) Relate the electric field at the point $(0, 0, z)$ produced by the infinite plane with the hole to your answers from part (a) and (b).

(d) Now that you know the electric field at the position of the point charge, calculate the force felt by the point charge.

(e) Describe the motion of the charge after we let go.

(a) The planar symmetry of the plane implies that the electric field can only point directly away or towards the plane (no other direction is unchanged after a reflection across *any* plane perpendicular to the charged plane). Therefore, we know that the electric field must point directly away from the plane if it is positive (as is the case in this problem). The planar symmetry also implies that the magnitude of the electric field does not depend on x and y; it can only possibly depend on the vertical distance from the charged plane. Therefore, $\mathbf{E}(\mathbf{x}) = E(z)\,\frac{z}{|z|}\,\hat{\mathbf{z}}$, where the factor $\frac{z}{|z|} = +1$ if $z > 0$ and -1 if $z < 0$ is only there to ensure that the direction of $\mathbf{E}$ is $\hat{\mathbf{z}}$ (directly up) if $z > 0$ and $-\hat{\mathbf{z}}$ (directly down) if $z < 0$. Now, we must choose a Gaussian surface. Let us choose a cylinder of height $2z$ and circular cross-sectional area A and place it such that the charged plane cuts it exactly in half, as indicated in the diagram below.



This satisfies the conditions for a good Gaussian surface because the magnitude of the electric field is constant over the top and bottom caps and the angle between $\mathbf{E}$ and $d\mathbf{a}$ is also constant (namely, 0) over the top and bottom caps (recall that $d\mathbf{a}$ is the vector whose magnitude is da, a differential bit of area on the Gaussian surface, and whose direction is perpendicular to the surface pointing outward). With the information we have at the moment, the magnitude of the electric field varies across the curved surface of the cylinder (I will call the cylinder a "soupcan" and the curved surface the "label"). Why is this not a problem? Well, because $\mathbf{E}$ does not actually pierce the "label" and thus there is no electric flux over the "label". That is, $\mathbf{E}$ is perpendicular to $d\mathbf{a}$ over the "label".

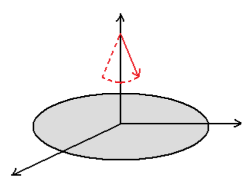
Now, Gauss' law tells us

$$\frac{\sigma A}{\epsilon_0} = \frac{Q_{\text{enc}}}{\epsilon_0} = \left(\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{label}}\right)\mathbf{E}\cdot d\mathbf{a} = E(z)\left(\int_{\text{top}} + \int_{\text{bottom}}\right)d\mathbf{a} + 0 = 2E(z)\,A \tag{1.1}$$

Therefore, the magnitude of the electric field for the infinite plane is $E(z) = \sigma/2\epsilon_0$. The electric field is therefore

$$\boxed{\mathbf{E}(\mathbf{x}) = \frac{\sigma}{2\epsilon_0}\,\frac{z}{|z|}\,\hat{\mathbf{z}}} \tag{1.2}$$

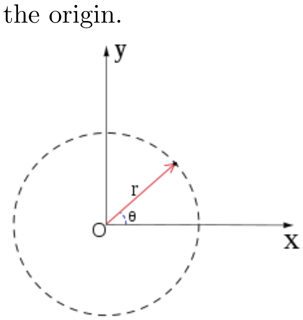
(b) The disk has rotational symmetry around the z-axis. That is, the disk is left unchanged if we rotate about the z-axis. Such a rotation does not move any of the points on the z-axis. These two things together imply that the electric field produced by the disk at a point on the z-axis must be unchanged if we rotate about the z-axis. If the field vector points off the z-axis as drawn below, then it is changed by a rotation about the z-axis. In fact, the collection of vectors produced under all rotations about the z-axis is a cone, as drawn. It follows that the electric field on the z-axis must point directly along the z-axis. It must point towards the disk since the disk has negative charge density.



The magnitude of the electric field on the z-axis can only possibly be a function of z (since x and y are zero there). Therefore, the form of the electric field on the z-axis is $\mathbf{E}((0,0,z)) = -E(z)\,\frac{z}{|z|}\,\hat{\mathbf{z}}$ (again, the $-\frac{z}{|z|}$ part just ensures that the field always points towards the disk).

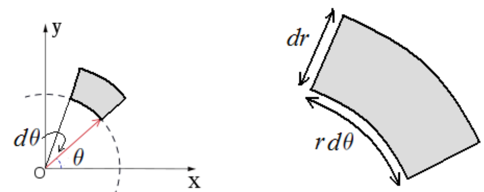
We cannot use Gauss' law here because the disk has neither spherical nor cylindrical nor planar symmetry. We don't have a quantitative expression even for the direction of the electric field away from the z-axis and thus we haven't the slightest clue what a good Gaussian surface would be. Thus, we have to use Coulomb's law and superposition. We must break the disk up into very many differential bits of surface area, concentrate the differential bit of charge contained in that area at a point in this area, compute the differential bit of electric field produced by that charge on the z axis, and integrate over the entire disk.

Since we are working with a disk, it would be wise to use circular coordinates for the x-y plane. Let θ be the angle from the positive x-axis increasing in the counter-clockwise direction. Let r be the radial distance in the x-y plane from the origin.



If you want to know how to retrieve the old x and y coordinates, then $x = r\cos\theta$ and $y = r\sin\theta$. The inverse is $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$.

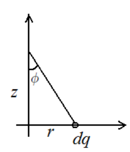
Consider a piece of area like the one shaded in the diagram below. The right hand diagram is a zoomed-in picture.



Recall that the arclength of the inner curved edge is $rd\theta$ since it is at a radial distance of r and subtends an angle $d\theta$ (note: $d\theta$ must be in radians here). It may seem a bit strange to approximate the area of the shaded region as $da = (rd\theta)(dr) = r\,dr\,d\theta$ since it is not a rectangle of those sidelengths. However, we are supposed to take $d\theta$ and dr to be infinitesimally small and in that limit, it is essentially a rectangle. Then the charge contained in da is $dq = -\sigma\,da = -\sigma r\,dr\,d\theta$. We concentrate this charge at the corner of the differential area with circular coordinates (r,θ) (in the shaded region in the diagram, that would be the bottom left corner).

This dq will contribute some $d\mathbf{E}$ electric field at the point $(0,0,z)$ on the z-axis. However, this field will not point along the z-axis, it will in fact point towards dq. We only want the z-component of it since all other components must cancel out if we integrate over the whole disk.

The distance between dq at circular coordinates (r,θ) and the point $(0,0,z)$ on the z-axis is $\sqrt{z^2+r^2}$. Define the angle ϕ such that $\cos\phi = \frac{|z|}{\sqrt{z^2+r^2}}$ as in the diagram below (note: the absolute value is there to deal with possible negative values of z),



Then the z-component of the differential bit of electric field produced at the point $(0,0,z)$ by dq at circular position (r,θ) is

$$dE_z = \frac{|dq|}{4\pi\epsilon_0(z^2+r^2)}\cos\phi = \frac{\sigma r\,dr\,d\theta}{4\pi\epsilon_0(z^2+r^2)}\,\frac{|z|}{\sqrt{z^2+r^2}} = \frac{\sigma|z|}{4\pi\epsilon_0}\,\frac{r\,dr}{(z^2+r^2)^{3/2}}\,d\theta \tag{1.3}$$

Note that I put absolute values around dq because I am only computing the magnitude of the electric field; I already know the direction. Then, the whole electric field magnitude is

$$\begin{aligned} E(z) = \frac{\sigma|z|}{4\pi\epsilon_0}\int_0^R \frac{r\,dr}{(z^2+r^2)^{3/2}}\int_0^{2\pi} d\theta &= \frac{\sigma|z|}{2\epsilon_0}\,\frac{1}{\sqrt{z^2+r^2}}\Bigg|_0^R \\ &= \frac{\sigma}{2\epsilon_0}\left(1-\frac{|z|}{\sqrt{z^2+R^2}}\right) \end{aligned} \tag{1.4}$$

Note that $\sqrt{z^2} = |z|$. Then, the electric field vector is

$$\boxed{\mathbf{E}(z) = -\frac{\sigma}{2\epsilon_0}\left(1-\frac{|z|}{\sqrt{z^2+R^2}}\right)\frac{z}{|z|}\,\hat{\mathbf{z}} = -\frac{\sigma}{2\epsilon_0}\,\frac{z}{|z|}\,\hat{\mathbf{z}} + \frac{\sigma z}{2\epsilon_0\sqrt{z^2+R^2}}\,\hat{\mathbf{z}}} \tag{1.5}$$

(c) If we superpose the infinite plane with charge density σ and the disk with charge density $-\sigma$, then the charge density within the region of the disk cancels out and we effectively get a positive infinite plane with a hole! Therefore, the electric field of the plane with the hole is just the sum of our answers from part (a) and (b),

$$\boxed{\mathbf{E}(z) = \frac{\sigma z}{2\epsilon_0\sqrt{z^2 + R^2}}\,\hat{\mathbf{z}}} \tag{1.6}$$

(d) The force felt by the point charge $-q$ on the z-axis is

$$\boxed{\mathbf{F}(z) = -q\mathbf{E}(z) = -\frac{q\sigma z}{2\epsilon_0\sqrt{z^2 + R^2}}\,\hat{\mathbf{z}}} \tag{1.7}$$

(e) When the charge is slightly above the center of the hole, then $z > 0$ and we see that the force, given by equation (1.7), points in the $-\hat{\mathbf{z}}$ direction, or down.

When the charge is slightly below the center of the hole, then $z < 0$ and we see that the force points in the $+\hat{\mathbf{z}}$ direction, or up.

Therefore, the point charge must $\boxed{\text{oscillate up and down}}$.

2 Capacitors and Dielectrics

A parallel-plate capacitor has square plates of side length $L = 10$ cm and a separation of $d = 4$ mm. For the following, let the dielectric have constant $\kappa = 2$. Compute the capacitance in the following scenarios:

(a) No dielectric. Call this C_0.

(b) Filled with dielectric. Call this C_1.

(c) Half-filled with a 10 cm × 10 cm × 2 mm slab of dielectric. Call this C_2.

(d) Half-filled with a 10 cm × 5 cm × 4 mm slab of dielectric. Call this C_3.

(e) Show that configuration (d) is at least "as good as" configuration (c).

(a) Evenly spread a charge Q on one plate and $-Q$ on the other so that the charge density (on the positive plate) is constant and given by $\sigma_0 = Q/A$. Then, the electric field magnitude between the plates is $E_0 = \sigma_0/\epsilon_0 = Q/\epsilon_0 A$. The potential difference between the plates is $V_0 = E_0 d = Qd/\epsilon_0 A$. Therefore, the capacitance is

$$\boxed{C_0 = \frac{Q}{V_0} = \frac{\epsilon_0 A}{d} = \frac{(8.9 \times 10^{-12}\ \text{F/m})(10^{-1}\ \text{m})^2}{4 \times 10^{-3}\ \text{m}} = 2.23 \times 10^{-11}\ \text{F} = 22.3\ \text{pF}} \tag{2.1}$$

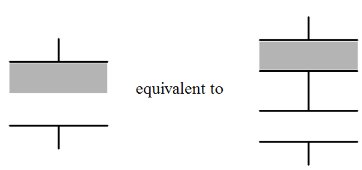
(b) Recall that the dielectric builds up a little bit of negative charge at its boundary that is closest to the positive plate and it builds up some positive charge at its boundary that is closest to the negative plate. These "induced" charge densities at the upper and lower boundaries of the dielectric produce an "induced" electric field that points opposite to the field produced by the parallel plates alone. Thus, the dielectric tends to *decrease* the magnitude of the electric field between the plates by a factor given by its dielectric constant.

Following everything we did in part (a), the only difference is that $E = E_0/\kappa$ and so $V = V_0/\kappa$. Therefore, the capacitance is

$$\boxed{C_1 = \frac{Q}{V_0/\kappa} = \kappa C_0 = 44.6\ \text{pF}} \tag{2.2}$$

Method 1 (my preferred method):

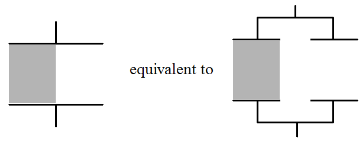
(c) We can treat this as two capacitors in series each having half the separation distance and one being filled with dielectric:



The capacitance of the bottom unfilled half is $C_2' = \frac{\epsilon_0 A}{d/2} = 2C_0$ and the upper filled half is $C_2'' = \frac{\kappa\epsilon_0 A}{d/2} = 2\kappa C_0$. Therefore, the equivalent capacitance is

$$\boxed{C_2 = \frac{1}{\frac{1}{C_2'} + \frac{1}{C_2''}} = \frac{1}{\frac{1}{2C_0} + \frac{1}{2\kappa C_0}} = C_0\left(\frac{2\kappa}{\kappa + 1}\right) = 29.7\ \text{pF}} \tag{2.3}$$

(d) We can treat this as two capacitors in parallel each having half the area and one being filled with dielectric:



The capacitance of the right unfilled half is $C_3' = \frac{\epsilon_0 (A/2)}{d} = \frac{C_0}{2}$ and the left filled half is $C_3'' = \frac{\kappa\epsilon_0 (A/2)}{d} = \frac{\kappa C_0}{2}$. Therefore, the equivalent capacitance is

$$\boxed{C_3 = C_3' + C_3'' = \frac{C_0}{2} + \frac{\kappa C_0}{2} = C_2\left(\frac{\kappa+1}{2}\right) = 33.4\text{ pF}} \tag{2.4}$$

Method 2 (this works too):

(c)' The gap where there is no dielectric has width $d/2$. In here, the electric field is the same as it is in part (a): $E_{\text{gap}} = E_0 = Q/\epsilon_0 A$ where Q is the fiducial charge on the positive plate. Thus, the potential difference across the gap is $V_{\text{gap}} = E_{\text{gap}} d/2 = E_0 d/2 = V_0/2$ where $V_0 = E_0 d$ is the potential difference between the plates in configuration (a). The electric field in the dielectric is $E_{\text{slab}} = E_0/\kappa$ and thus $V_{\text{slab}} = E_{\text{slab}} d/2 = E_0 d/2\kappa = V_0/2\kappa$. The total potential difference is just the sum: $V = V_{\text{gap}} + V_{\text{slab}} = V_0\left(\frac{1+(1/\kappa)}{2}\right) = V_0\left(\frac{\kappa+1}{2\kappa}\right)$. Therefore, the capacitance is

$$\boxed{C_2 = \frac{Q}{V_0\left(\frac{\kappa+1}{2\kappa}\right)} = C_0\left(\frac{2\kappa}{\kappa+1}\right) = 29.7\text{ pF}} \tag{2.5}$$

(d)' Let V be the potential difference across the plates (which is the same whether we are in the gap or dielectric region). The field in the gap region is $E_{\text{gap}} = V/d$ and so the charge density in the portion of the plate above the gap is $\sigma_{\text{gap}} = \epsilon_0 E_{\text{gap}} = \epsilon_0 V/d$. Since the potential across the gap is the same as that across the dielectric slab, the electric field must be the same as well: $E_{\text{slab}} = \epsilon_0 V/d$. This means that more charge can be stored in the plates above and below the dielectric: $\sigma_{\text{slab}} = \kappa\sigma_{\text{gap}} = \kappa\epsilon_0 V/d$. Thus, the capacitance is

$$\boxed{C_3 = \frac{(\sigma_{\text{gap}} + \sigma_{\text{slab}})(A/2)}{V} = \frac{\epsilon_0 A}{d}\left(\frac{\kappa+1}{2}\right) = C_0\left(\frac{\kappa+1}{2}\right) = 33.4\text{ pF}} \tag{2.6}$$

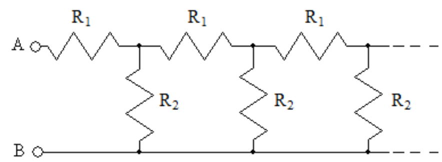
(e) By "better", we mean the one with higher capacitance since it is able to store more charge with less electric field and thus less energy. Of course, (b) is always better than all of the other choices. Our claim is that (d) is at least as good as (c). This is true because $\kappa \geq 1$ (recall that the dielectric constant of vacuum is 1 and of any other substance is higher than that). Therefore,

$$(\kappa-1)^2 \geq 0 \quad\Longrightarrow\quad \kappa^2 - 2\kappa + 1 \geq 0 \quad\Longrightarrow\quad (\kappa+1)^2 \geq 4\kappa \quad\Longrightarrow\quad \boxed{\frac{\kappa+1}{2} \geq \frac{2\kappa}{\kappa+1}} \tag{2.7}$$

It follows that $\boxed{C_3 \geq C_2}$.

If you're wondering where I used the fact that $\kappa \geq 1$, it was in crossing the final double arrow in (2.5). I divided both sides of the inequality by $2(\kappa+1)$. Had $2(\kappa+1)$ been negative, the inequality sign would have been flipped and we would have gotten the exact opposite of what we wanted! Therefore, actually, all I need is that $\kappa > -1$ (strict inequality since if $\kappa = -1$, then we would be dividing by 0!). This is definitely the case since, in truth, $\kappa \geq 1$.

3 Infinite Chain of Resistors



(*Purcell* 4.32, page 167) Some important kinds of networks are infinite in extent. The figure above shows a chain of resistors stretching off endlessly to the right. This is sometimes called an attenuator chain, or a ladder network. We would like to find the "input resistance," that is, the equivalent resistance between terminals A and B.

(a) Call the equivalent resistance of this network R. Add one more chain on the left (one more R_1 branching off to the left from point A and one more R_2 between points A and B). Calculate the equivalent resistance R' of this new circuit.

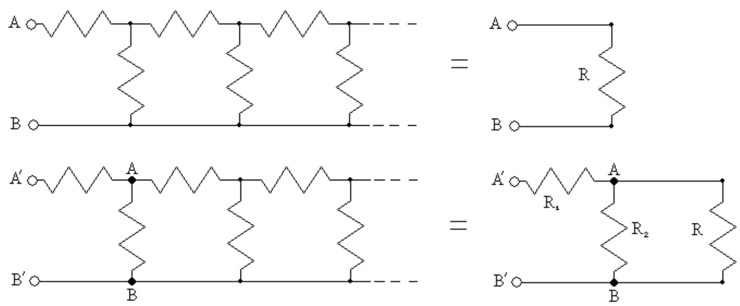
(b) Which of the following should be true: $R' > R$, $R' = R$ or $R' < R$? (Hint: have I changed the network by adding one more chain? If I have, how so?)

(c) Solve for R in terms of R_1 and R_2.

(d) Suppose I instead attach a battery of voltage V between terminals A and B. Call the voltage drop across the first (left-most) R_2 resistor V_1, then V_2 for the second R_2 and so on. Write V_n in terms of the voltage V of the battery, R_1, R_2 and R (the answer may or may not involve all of these).

(e) If I want the voltage V_n to be cut in half after each step along the chain (that is, $V_{n+1} = V_n/2$), then how must R_1 and R_2 be related and what is the equivalent resistance R of the chain in this case?

(a) Below is a diagram of the proposed extension



From the diagram, we have

$$R' = R_1 + \frac{R_2 R}{R + R_2} \tag{3.1}$$

(b) We have not actually changed the network because it continues infinitely off to the right - adding any finite number of chains on the left does not change anything. Therefore, we must have $\boxed{R' = R}$.

(c) We set $R' = R$ and we get a quadratic equation for R.

$$R' = R_1 + \frac{R_2 R}{R + R_2} = R \quad \Longrightarrow \quad R^2 - R_1 R - R_1 R_2 = 0 \tag{3.2}$$

We solve this quadratic and drop the negative solution since R cannot be negative!

$$\boxed{R = \frac{R_1 + \sqrt{R_1^2 + 4R_1R_2}}{2}} \tag{3.3}$$

(d) Note that the voltage dropped across the n^{th} R_2 resistor is also dropped across the remainder of the circuit to the right of that n^{th} R_2 resistor since all that stuff as a whole is in parallel with the n^{th} R_2 resistor. But the stuff to the right of the n^{th} R_2 resistor still has equivalent resistance R. Therefore, the current passing through to the right of the n^{th} node (which is the branching point above the n^{th} R_2 resistor) is $I_n = V_n/R$.

To find the voltage at the $(n+1)^{\text{st}}$ node, we subtract from V_n the voltage dropped across the one R_1 resistor between the n^{th} and $(n+1)^{\text{st}}$ nodes. Thus,

$$V_{n+1} = V_n - I_nR_1 = V_n\left(1 - \frac{R_1}{R}\right) \tag{3.4}$$

This is a geometric sequence. The voltage at the next node is simply the voltage at the node before it multiplied by some constant factor $1 - R_1/R$. Thus, the voltage at the n^{th} node, which is the voltage dropped across the n^{th} R_2 resistor is

$$\boxed{V_n = V\left(1 - \frac{R_1}{R}\right)^n} \tag{3.5}$$

(e) We need $1 - R_1/R = 1/2$ and so $\boxed{R = 2R_1}$ where R is given in (3.3). We find that $\boxed{R_1/R_2 = 1/2}$.

4 Moving Charges between Concentric Cylinders

Suppose I have two concentric conducting cylinders both of length L with the smaller one having radius a and the larger one having radius b. Suppose that $L >> b$ so that as far as the electric field is concerned, these are essentially infinitely long. These cylinders start out electrically neutral. How much work does it take to transfer charge from one to the other until they carry charges $\pm Q$? (Hint: review how you solved Problem 23.39 in problem set 3).

Method 1 (doing an integral): Suppose we are at the point where the cylinders carry charges $\pm q$. Suppose that the inner cylinder carries the positive charge $+q$. Then, the electric field between the two cylinders is given by Gauss's law, using a concentric Gaussian cylinder of length L and radius r (such that $a \leq r \leq b$). Then all the positive charge q on the inner cylinder is enclosed:

$$2\pi r L E = \int \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\epsilon_0 = q/\epsilon_0 \qquad \Longrightarrow \qquad E(q,r) = \frac{q}{2\pi\epsilon_0 L r} \tag{4.1}$$

We have written $E(q,r)$ as a function both of q and r since q will vary from 0 at the beginning to Q at the end.

Negative of the derivative of the potential $V(q,r)$ with respect to r must give me $E(q,r)$ given above. Therefore, the potential between the two cylinders must be

$$V(q,r) = -\frac{q}{2\pi\epsilon_0 L} \ln r + V_0 \tag{4.2}$$

where V_0 is just some arbitrary constant (it is fixed by imposing, for example, that the potential vanishes at infinity).

Therefore, the potential difference between the two cylinders is

$$V(q) \equiv V(q,a) - V(q,b) = -\frac{q}{2\pi\epsilon_0 L} \ln a + V_0 - \left(-\frac{q}{2\pi\epsilon_0 L} \ln b + V_0\right) = \frac{q}{2\pi\epsilon_0 L} \ln \frac{b}{a} \tag{4.3}$$

Now, we must move a charge dq (taken to be positive) from the more negative cylinder (with radius b) to the more positive cylinder (with radius a) to increase the magnitude of the charges on the cylinders from q to $q + dq$. How much work must I do here? Well, the differential bit of work I must do is

$$dW = dq\, V(q) = \frac{\ln(b/a)}{2\pi\epsilon_0 L}\, q\, dq \tag{4.4}$$

We must integrate dW to get the total work

$$\boxed{W = \int dW = \frac{\ln(b/a)}{2\pi\epsilon_0 L} \int_0^Q q\, dq = \frac{Q^2 \ln(b/a)}{4\pi\epsilon_0 L}} \tag{4.5}$$

Method 2 (energy conservation): Let us find the capacitance of the concentric cylinders. Well, we already know $V(q)$ from (4.3) and so the capacitance is simply

$$C = \frac{q}{V(q)} = \frac{2\pi\epsilon_0 L}{\ln(b/a)} \tag{4.6}$$

The energy stored in the capacitor at the beginning is 0. The work I did is thus the energy stored at the end when the cylinders carry charges $\pm Q$, which is

$$\boxed{\frac{1}{2}CV^2 = \frac{1}{2} \cdot \frac{2\pi\epsilon_0 L}{\ln(b/a)} \cdot \left(\frac{Q \ln(b/a)}{2\pi\epsilon_0 L}\right)^2 = \frac{Q^2 \ln(b/a)}{4\pi\epsilon_0 L}} \tag{4.7}$$

5 Expanding Loop in a Solenoid

Suppose we have a loop of conducting wire with resistance R having some initial area (note: the loop is not necessarily a circle; it could form any 2D (flat) shape). This loop is placed inside a solenoid of length L, which has N windings carrying current I_s. The loop always remains inside the solenoid and always lies flat inside the solenoid such that the plane of the loop is parallel with the planes of the windings of the solenoid. Suppose that I now somehow expand the loop of wire such that its area increases at the constant rate $\dot{A}$ (note: $\dot{A}$ is a constant having units m^2/s; also, the shape of the loop could very well be changing as long as the area is increasing at a constant rate; finally, I manage to do this without increasing the mass of the loop). I eventually stop after some time T has elapsed. How much work did I do? (Assume that the process of changing the area of the loop would not have cost any work had the loop not been in the magnetic field).

———————o———————

Recall that the magnetic field inside a solenoid is constant and its magnitude is given by $B = \mu_0 n I_s = \mu_0 N I_s / L$ where $n = N/L$ is the linear density of windings of the solenoid (how many windings in some unit length of the solenoid). The direction of the magnetic field is parallel to the axis (or perpendicular to the planes of the windings). Since the plane of the stretchy loop is parallel to the planes of the windings, the magnetic field is perpendicular to the plane of the loop and since it's constant in magnitude, the magnetic flux just becomes the product of the magnetic field magnitude and the area of the loop:

$$\Phi_B = \int \mathbf{B} \cdot d\mathbf{a} = B \int da = BA = \frac{\mu_0 N I_s A}{L} \tag{5.1}$$

The only thing changing over time is A and so the induced emf in the stretchy loop of wire is

$$|\mathcal{E}| = \left| \frac{d\Phi_B}{dt} \right| = \frac{\mu_0 N I_s}{L} \frac{dA}{dt} = \frac{\mu_0 N I_s \dot{A}}{L} \tag{5.2}$$

Then, the induced current is given by Ohm's law

$$I = \frac{|\mathcal{E}|}{R} = \frac{\mu_0 N I_s \dot{A}}{RL} \tag{5.3}$$

This current is constant (over time) since μ_0, N, I_s, $\dot{A}$, R and L are all constant. Then, the rate at which the loop is dissipating energy (through heat), or the power, is

$$P = I^2 R = \frac{\mu_0^2 N^2 I_s^2 \dot{A}^2}{RL^2} \tag{5.4}$$

Again, this power is constant (over time) and so we can pull it out of the time integral below. Therefore, the total energy dissipated by the loop is

$$\boxed{W = \int P\, dt = P \int dt = PT = \frac{\mu_0^2 N^2 I_s^2 \dot{A}^2 T}{RL^2}} \tag{5.5}$$

I have called this W to suggest that it is indeed the work I had to do to increase the area of the loop. This is a consequence of energy conservation. The loop has the same energy before the increase in area ($t < 0$) as it does afterwards ($t > T$): there is no kinetic energy, the mass of the loop has not changed (rather unrealistic, but oh well) and there is no induced current when $t < 0$ or $t > T$. Therefore, all the energy that I put in to the loop, which is the work I do to it, must have been dissipated away by the loop through heat (via a current flowing through its resistance). Hence, (5.5) is both the total energy dissipated by the loop AND the work I had to do to increase its area.

6 Challenge Question: Magnetic Monopoles

Note: I will probably not present this problem during the review since it is also probably considerably more challenging than a typical exam question. But, it's here as a challenge question.

The field lines we have seen do not start or end anywhere unlike electric field lines that started and ended on point charges. However, we can still imagine a point-like object (like a point-charge, just without the electric charge) that is a source (or sink) of magnetic field lines. I believe that the scientific concensus is that we have never experimentally observed such an object, but we can still imagine what would happen if one did exist. Such an object is called a magnetic monopole (because it is like an object just having one magnetic pole rather than both North and South poles).

A magnetic monopole (MM) produces a magnetic field like the electric field of a point charge:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 g}{4\pi r^2}\,\hat{\mathbf{r}} \tag{6.1}$$

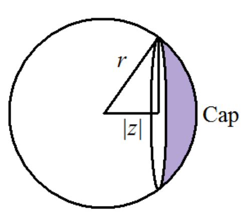
where g measures the magnetic "charge" of the MM (the magnetic analogue of electric charge) and where r is the radial distance from the position of the MM.

Imagine that we place a circular loop of wire of resistance R and radius R_ℓ in the x-y plane centered at the origin and that a MM with magnetic charge $g > 0$ travels along the z-axis from $z = -\infty$ (starting at time $t = -\infty$) to $z = +\infty$ (ending at time $t = +\infty$) and it passes through the loop at $t = 0$. The speed of the MM is v and is constant and much less than the speed of light (weird things would happen if it were moving too fast).

(a) Draw a picture of the situation when the MM is at position $z < 0$ and another one for $z > 0$. In both cases, draw the direction in which the MM is moving and draw the magnetic field lines of the MM making sure to depict which field lines contribute to the magnetic flux through the loop of wire.

(b) At any given moment, imagine freezing time when the MM is at position z (positive or negative). Imagine a sphere, S, centered at the MM and having radius equal to the distance between the MM and any point on the wire loop. Carefully draw this sphere S into one of your drawings from part (a). Compute the magnetic flux, Φ_S through the whole sphere S.

(c) The loop divides S into two pieces, S_1 and S_2 with S_1 having the generically smaller surface area of the two. We will call S_1 the "cap". Label these pieces in your diagram. Let A and A_1 be the surface area of S and S_1, respectively. Write A and A_1, which are functions of the position, z, of the MM along the z-axis, in terms of things given in the problem. (Note: you will need the formula for the surface area of a "cap" of a sphere in terms of the radius of the sphere, r, and the distance, $|z|$, from the center of the sphere to the center of the circle that bounds that cap. This is $A_{\text{cap}} = 2\pi r^2(1 - \frac{|z|}{r})$.)



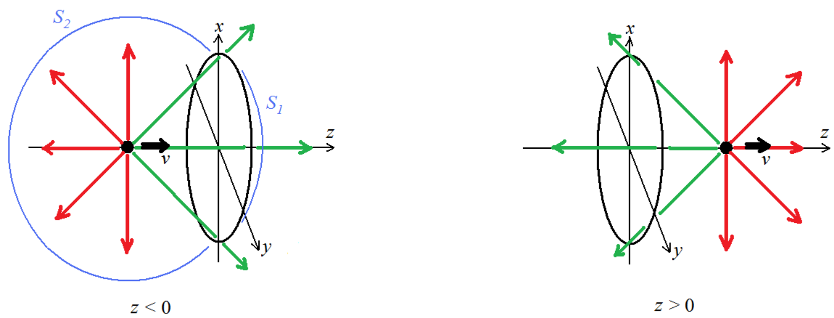
(d) Express $|\Phi_{S_1}|$, the magnitude of the magnetic flux through S_1, in terms of Φ_S, A and A_1. Use your answer to part (c) to write $|\Phi_{S_1}|$ in terms of things given in the problem.

(e) What is the relationship between $|\Phi_{\text{loop}}|$, the magnitude of the magnetic flux through the loop, and $|\Phi_{S_1}|$?

(f) Calculate the magnitude of the induced emf in the wire as a function of time, $|\mathcal{E}_{\text{ind}}(t)|$.

(g) Calculate the induced current in the wire as a function of time, $I(t)$. If you looked at the loop from above (i.e. such that the MM is moving directly towards you), in which direction would the induced current be flowing in the loop?

(a) and (b) The black thick arrow shows the direction of motion of the MM. The green and red lines are the magnetic field lines, which look exactly the same as the electric field lines of a positive point charge. The red lines are field lines that do not contribute to the magnetic flux through the loop and the green field lines are those that do contribute. The sphere S is drawn in blue (the sphere grazes the loop). The smaller area subtended by the loop is S_1.



Let r be the distance between the MM and a point on the loop. Since the magnetic field is constant in magnitude along the sphere and is always perpendicular to the surface (or parallel to $d\mathbf{a}$), the magnetic flux is simply given by the product of the magnetic field and the surface area of the sphere:

$$\boxed{\Phi_S = \int_S \mathbf{B} \cdot d\mathbf{a} = BA = \frac{\mu_0 g}{4\pi r^2} \cdot 4\pi r^2 = \mu_0 g} \tag{6.2}$$

(c) The distance between the MM and a point on the loop is

$$r = \sqrt{z^2 + R_\ell^2} \tag{6.3}$$

Therefore, the area of the whole sphere S is

$$\boxed{A = 4\pi r^2 = 4\pi(z^2 + R_\ell^2)} \tag{6.4}$$

We were given the formula for the area of the cap:

$$\boxed{A_1 = 2\pi r^2 \left(1 - \frac{|z|}{r}\right) = 2\pi(z^2 + R_\ell^2)\left(1 - \frac{|z|}{\sqrt{z^2 + R_\ell^2}}\right)} \tag{6.5}$$

(d) As we mentioned in part (b), since the magnetic field is always parallel to $d\mathbf{a}$ on the sphere, the flux is simply given by the magnitude of the field multiplied by the surface area. Therefore,

$$\boxed{|\Phi_{S_1}| = BA_1 = \frac{BAA_1}{A} = \frac{\Phi_S A_1}{A} = \frac{\mu_0 g \cdot 2\pi r^2}{4\pi r^2}\left(1 - \frac{|z|}{r}\right) = \frac{\mu_0 g}{2}\left(1 - \frac{|z|}{\sqrt{z^2 + R_\ell^2}}\right)} \tag{6.6}$$

(e) All the field lines that pierce the area inside the loop, eventually pass through and pierce the surface S_1. It follows that

$$\boxed{|\Phi_{\text{loop}}| = |\Phi_{S_1}| = \frac{\mu_0 g}{2}\left(1 - \frac{|z|}{\sqrt{z^2 + R_\ell^2}}\right)} \tag{6.7}$$

(f) We are told that the speed is v and is constant. Therefore, $z = vt + z_0$ where z_0 is z at time $t = 0$. We are told that the MM passes through the loop at $t = 0$ and so $z_0 = 0$ and we have $z = vt$. Therefore,

$$\begin{aligned}
|\mathcal{E}_{\text{ind}}| = \left|\frac{d\Phi_{\text{loop}}}{dt}\right| &= \left|\frac{d\Phi_{\text{loop}}}{dz}\cdot\frac{dz}{dt}\right| = v\left|\frac{d\Phi_{\text{loop}}}{dz}\right| \\
&= v\left| -\frac{\mu_0 g}{2}\cdot\frac{\sqrt{z^2+R_\ell^2}\,\frac{|z|}{z} - |z|\,\frac{z}{\sqrt{z^2+R_\ell^2}}}{z^2+R_\ell^2}\right| \\
&= \frac{\mu_0 g v}{2(z^2+R_\ell^2)^{3/2}}\left|(z^2+R_\ell^2)\frac{|z|}{z} - z|z|\right| \\
&= \boxed{\frac{\mu_0 g v R_\ell^2}{2(z^2+R_\ell^2)^{3/2}}}
\end{aligned} \tag{6.8}$$

where we get the second line by the rule for differentiating quotients and we used the fact that $\frac{d|z|}{dz} = +1$ if $z > 0$ and -1 if $z < 0$ and so $\frac{d|z|}{dz} = \frac{|z|}{z}$. To get the third line, we multiplied the total numerator and denominator by $\sqrt{z^2 + R_\ell^2}$. The first and last terms in the big || signs of the third line cancel (i.e. $z^2\frac{|z|}{z} - z|z| = z|z| - z|z| = 0$). Finally, $|\frac{|z|}{z}| = 1$ and we get the final line.

(g) The induced current is

$$\boxed{I_{\text{ind}} = \frac{|\mathcal{E}_{\text{ind}}|}{R} = \frac{\mu_0 g v R_\ell^2}{2R(z^2+R_\ell^2)^{3/2}}} \tag{6.9}$$

From the top view, when the MM is approaching the loop from negative z, the magnetic field lines come out of the page through the loop and the magnetic field is increasing in magnitude. Therefore, the magnetic flux out of the page through the loop is increasing. Therefore, the induced current in the wire must flow in such a direction as to produce an induced magnetic field that points into the page inside the loop. That current would be clockwise.

When the MM is leaving the loop at positive z, the magnetic field lines go into the page through the loop and the magnetic field is decreasing in magnitude. Therefore, the magnetic flux into the page through the loop is decreasing. This is essentially equivalent to magnetic flux out of the page through the loop increasing. Therefore, the induced current is still clockwise.

The induced current flows in the $\boxed{\text{clockwise}}$ direction as viewed from above.