Midterm 2 Review Problems
Physics 8B
Fall 2009

Complex number review

AC circuits are usually handled with one of two techniques: phasors and complex numbers. We'll be using the complex number approach, so first we'll go over the basics of complex numbers. This won't be covered at the review, so if you're shaky on complex numbers, you might want to review this ahead of time.
The fundamental quantity in complex numbers is $i = \sqrt{-1}$. By adding this number to the normal, real numbers, we obtain an entirely consistent, and very useful number system. You can add, subtract, multiply, and divide using i just like normal numbers; it can even be put in an exponential, like $e^{i\pi}$.
Using imaginary numbers, we expand the numbers we can use from calculation; now, we can use regular numbers (like 1, 5.2, 42, etc.), purely complex numbers (like i, $6i$, $1.2i$), and combinations of both (like $1 + i$, $5 - 2i$).

Adding and subtracting

When adding and subtracting with i, keep the real and imaginary parts separate. One way to think of this is to treat i like you would treat a variable x in a polynomial equation. For example, $(1+2i)+(-4+6i) = -3 + 8i$, just like $(1 + 2x) + (-4 + 6x) = -3 + 8x$.

Exercises:

a. Let $a = 4 + 9.2i$ and $b = 7.4 - 0.4i$. What is $a + b$? What is $a - b$?

Multiplying and dividing

When multiplying and dividing, you use all of the normal tricks of multiplying and dividing, like the distributive rule. The most important thing is that since $i = \sqrt{-1}$, $i^2 = -1$. It's best to see some examples:

$$\begin{aligned}
2 \times (1 + i) &= 2 + 2i \\
i \times (5 - 4i) &= 5i - 4i^2 = 4 + 5i \\
(1 + 2i) \times (4 - 2i) &= 4 - 2i + 8i - 4i^2 = 4 + 6i + 4 = 8 + 6i \\
(5 - 3i) \times (2 + i) &= 10 - 6i + 5i - 3i^2 = 10 - i + 3 = 13 - i \\
(a + bi) \times (c + di) &= ac + ibc + iad + i^2bd = (ac - bd) + i(bc + ad)
\end{aligned}$$

Dividing is the tricky part, since you may well end up with a complex number in the denominator. In general, this is not a good thing; we want to be able to write a complex number as (real part)+$i\times$(imaginary part), separating out the real and imaginary parts. The good news is that there's a simple recipe; the bad news is that the math can get complicated.
We'll do an example. Let $a = 5 + i$ and $b = 2 + 4i$. We want to compute $c = \frac{a}{b}$. This will give us some nasty imaginary numbers in the denominator:

$$c = \frac{a}{b} = \frac{5+i}{2+4i}$$

The trick is to multiply the top and the bottom by the *complex conjugate* of b. To get the complex conjugate of a number, just replace i with $-i$ everywhere you see it. So, the complex conjugate of b, which we'll write b^*, is $b^* = 2 - 4i$. Then, note that $bb^* = (2 + 4i)(2 - 4i) = 4 + 8i - 8i - 16i^2 = 4 + 16 = 20$. All of the imaginary parts have disappeared!

Let's multiply c by $\frac{b^*}{b^*}$ and see what happens.

$$\begin{aligned} c \times \frac{b^*}{b^*} &= \frac{5+i}{2+4i} \times \frac{2-4i}{2-4i} \\ &= \frac{(5+i)(2-4i)}{(2+4i)(2-4i)} \\ &= \frac{10+2i-20i-4i^2}{4+8i-8i-16i^2} \\ &= \frac{14-18i}{20} \\ &= \frac{7}{10} - i\frac{9}{10} \end{aligned}$$

So, we've gotten rid of all the nasty factors of i in the denominator, and wrote $c = \frac{a}{b}$ in the form (real part)+i(imaginary part)
One last thing: what is $\frac{1}{i}$? Just multiply the top and the bottom by i, to get $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$.

Exercises:

a. What is $2 \times (4-i)$? What is $\frac{8i-4}{2}$? What is $\frac{5+4i}{i}$?

b. Let $a = 3-7i$ and $b = 9+3i$. What is $a \times b$? What is $\frac{a}{b}$? Remember to write your answer in the form (real part)+i(imaginary part).

Exponents

Now for the interesting part. What happens when you put i in an exponent? This may seem like magic, or just a trick, but you can always write the following: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. If this seems strange, you can look it up in a math textbook, but this is something that every physicist and mathematician holds dear to his or her heart. It's very, very useful, and very elegant.
So, something like $e^{i\pi}$ will turn into $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$. Or, $e^{i\pi/2} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$.

Exercises:

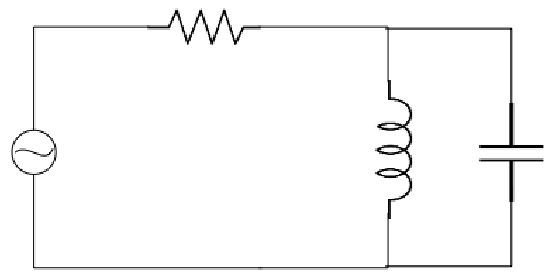
a. What is $e^{i\pi/4}$? More abstractly, what is e^{iwt}?

Last word

You might run into combinations of divisions and exponents. For example, you might run into something like $\frac{2e^{i\theta}}{3i}$. Any of the techniques we used above will lead to the correct answer. One way is as follows: we can write $i = 0 + 1i = cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = e^{i\pi/2}$, so $\frac{2e^{i\theta}}{3i} = \frac{2e^{i\theta}}{3e^{i\pi/2}} = \frac{2}{3}e^{i\theta - i\pi/2} = e^{i(\theta-\pi/2)}$.
Just remember, you're doing all the normal things that you do with real numbers (+, −, ×, /, exponents), but you have to keep track of your $i's$, just like when dealing with polynomials, you have to keep track of your x's and y's, and whenever you see i^2, you can replace it with -1.

Problem 1: AC Circuits

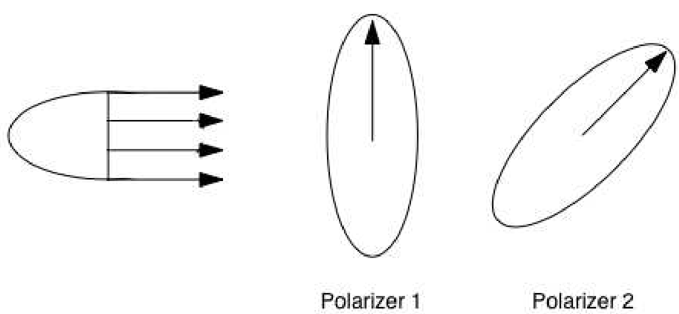
Consider the circuit below. The power supply provides a voltage $V(t) = V_0 \cos(\omega t)$, with $V_0 = 12$V. The inductor has inductance $L = 5$H, the capacitor has capacitance $C = 3\mu$F, and the resistor has resistance $R = 6$MΩ.



1. What is the total complex impedance Z?
2. Using the answer from part (a), what is the current $I(t)$ going through the power supply?
3. What is I_{rms}? What is V_{rms}?
4. What is the phase difference between $V(t)$ and $I(t)$?

Problem 2: Polarization

We start with a lamp shining unpolarized light in the $\hat{z}$ direction, with intensity 1.5$\frac{\text{W}}{\text{m}^2}$. The light passes through two polarizers, as shown in the diagram. The first polarizer is oriented in the $\hat{x}$ direction, and the second polarizer is oriented at a 45° angle, pointing in the $\frac{\hat{x}+\hat{y}}{\sqrt{2}}$ direction.



1. After the light passes through the first polarizer, how much intensity is transmitted?
2. (a) gives you the magnitude of the *Poynting vector* after the first polarizer, $|\vec{S}_1|$; this is a vector which tells you the intensity of your light and the direction the light is traveling in. You also know the polarization of the light. So, what is the magnitude and direction of $\vec{E}_1$, the transmitted electric field?
3. After the light hits the second polarizer, what is the final intensity of the light? What is the magnitude and direction of the electric field, $\vec{E}_2$?

Problem 3: Single lenses

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$$

1. A warmup problem. Suppose you have a concave mirror, with radius $r = 20\text{cm}$. What is the focal length f of the mirror? How would this be difference if the mirror were convex? Now suppose you have an object sitting o =25cm away from the mirror. Where is the image, i.e. what is i?

2. Ray tracing: draw the appropriate ray tracing diagram for part (a).

3. Diverging lenses: suppose you have a diverging lens with $f = -25\text{cm}$. Place an object of height $h_o = 5\text{cm}$, 40cm away from the lens. First, draw the ray tracing diagram. Then, find the image distance, the image height, and say whether the image is real or virtual.

4. Converging lenses: suppose you have a converging lens with $f = 15\text{cm}$. Place an object of height $h_0 = 3\text{cm}$ at $o = 10\text{cm}$. Where is the image? How large is the image? Now place the object at $o = 30\text{cm}$. Where is the image? How large is the image?

5. A plano-convex lens with radius $r = 29\text{cm}$ is made from glass with a refractive index dependent on the wavelength of light. Suppose a violet lamp ($\lambda \approx 420\text{nm}$) placed at $o = 40\text{cm}$ creates an image at $i = 17.85\text{cm}$, and a red lamp ($\lambda \approx 680\text{nm}$) placed at o =40cm creates an image at $i = 15.9\text{cm}$. What is the range of the index of refraction n of the lens? (I.e. what is $n_{red} - n_{violet}$?)

6. For fun: why won't this plano-convex lens work to focus x-rays?