

Midterm 2 Review Problems
Physics 8B
Fall 2009

Complex number review

AC circuits are usually handled with one of two techniques: phasors and complex numbers. We'll be using the complex number approach, so first we'll go over the basics of complex numbers. This won't be covered at the review, so if you're shaky on complex numbers, you might want to review this ahead of time.

The fundamental quantity in complex numbers is $i = \sqrt{-1}$. By adding this number to the normal, real numbers, we obtain an entirely consistent, and very useful number system. You can add, subtract, multiply, and divide using i just like normal numbers; it can even be put in an exponential, like $e^{i\pi}$.

Using imaginary numbers, we expand the numbers we can use from calculation; now, we can use regular numbers (like 1, 5.2, 42, etc.), purely complex numbers (like i , $6i$, $1.2i$), and combinations of both (like $1 + i$, $5 - 2i$).

Adding and subtracting

When adding and subtracting with i , keep the real and imaginary parts separate. One way to think of this is to treat i like you would treat a variable x in a polynomial equation. For example, $(1 + 2i) + (-4 + 6i) = -3 + 8i$, just like $(1 + 2x) + (-4 + 6x) = -3 + 8x$.

Exercises:

a. Let $a = 4 + 9.2i$ and $b = 7.4 - 0.4i$. What is $a + b$? What is $a - b$?

Multiplying and dividing

When multiplying and dividing, you use all of the normal tricks of multiplying and dividing, like the distributive rule. The most important thing is that since $i = \sqrt{-1}$, $i^2 = -1$. It's best to see some examples:

$$\begin{aligned}2 \times (1 + i) &= 2 + 2i \\i \times (5 - 4i) &= 5i - 4i^2 = 4 + 5i \\(1 + 2i) \times (4 - 2i) &= 4 - 2i + 8i - 4i^2 = 4 + 6i + 4 = 8 + 6i \\(5 - 3i) \times (2 + i) &= 10 - 6i + 5i - 3i^2 = 10 - i + 3 = 13 - i \\(a + bi) \times (c + di) &= ac + ibc + iad + i^2bd = (ac - bd) + i(bc + ad)\end{aligned}$$

Dividing is the tricky part, since you may well end up with a complex number in the denominator. In general, this is not a good thing; we want to be able to write a complex number as (real part)+ i ×(imaginary part), separating out the real and imaginary parts. The good news is that there's a simple recipe; the bad news is that the math can get complicated.

We'll do an example. Let $a = 5 + i$ and $b = 2 + 4i$. We want to compute $c = \frac{a}{b}$. This will give us some nasty imaginary numbers in the denominator:

$$c = \frac{a}{b} = \frac{5 + i}{2 + 4i}$$

The trick is to multiply the top and the bottom by the *complex conjugate* of b . To get the complex conjugate of a number, just replace i with $-i$ everywhere you see it. So, the complex conjugate of b , which we'll write b^* , is $b^* = 2 - 4i$. Then, note that $bb^* = (2 + 4i)(2 - 4i) = 4 + 8i - 8i - 16i^2 = 4 + 16 = 20$. All of the imaginary parts have disappeared!

Let's multiply c by $\frac{b^*}{b^*}$ and see what happens.

$$\begin{aligned}c \times \frac{b^*}{b^*} &= \frac{5+i}{2+4i} \times \frac{2-4i}{2-4i} \\&= \frac{(5+i)(2-4i)}{(2+4i)(2-4i)} \\&= \frac{10+2i-20i-4i^2}{4+8i-8i-16i^2} \\&= \frac{14-18i}{20} \\&= \frac{7}{10} - i\frac{9}{10}\end{aligned}$$

So, we've gotten rid of all the nasty factors of i in the denominator, and wrote $c = \frac{a}{b}$ in the form (real part)+ i (imaginary part)

One last thing: what is $\frac{1}{i}$? Just multiply the top and the bottom by i , to get $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$.

Exercises:

a. What is $2 \times (4 - i)$? What is $\frac{8i-4}{2}$? What is $\frac{5+4i}{i}$?

b. Let $a = 3 - 7i$ and $b = 9 + 3i$. What is $a \times b$? What is $\frac{a}{b}$? Remember to write your answer in the form (real part)+ i (imaginary part).

Exponents

Now for the interesting part. What happens when you put i in an exponent? This may seem like magic, or just a trick, but you can always write the following: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$. If this seems strange, you can look it up in a math textbook, but this is something that every physicist and mathematician holds dear to his or her heart. It's very, very useful, and very elegant.

So, something like $e^{i\pi}$ will turn into $e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1$. Or, $e^{i\pi/2} = \cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}) = i$.

Exercises:

a. What is $e^{i\pi/4}$? More abstractly, what is $e^{i\omega t}$?

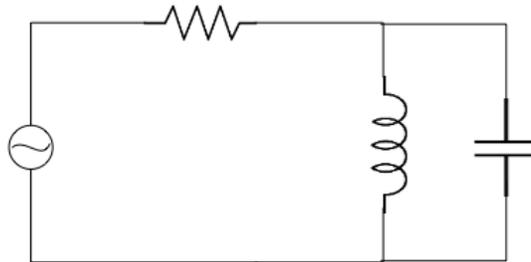
Last word

You might run into combinations of divisions and exponents. For example, you might run into something like $\frac{2e^{i\theta}}{3i}$. Any of the techniques we used above will lead to the correct answer. One way is as follows: we can write $i = 0 + 1i = \cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}) = e^{i\pi/2}$, so $\frac{2e^{i\theta}}{3i} = \frac{2e^{i\theta}}{3e^{i\pi/2}} = \frac{2}{3}e^{i\theta-i\pi/2} = e^{i(\theta-\pi/2)}$.

Just remember, you're doing all the normal things that you do with real numbers (+, -, ×, /, exponents), but you have to keep track of your i 's, just like when dealing with polynomials, you have to keep track of your x 's and y 's, and whenever you see i^2 , you can replace it with -1 .

Problem 1: AC Circuits

Consider the circuit below. The power supply provides a voltage $V(t) = V_0 \cos(\omega t)$, with $V_0 = 12\text{V}$ and $\omega = 120\text{Hz}$. The inductor has inductance $L = 5\text{H}$, the capacitor has capacitance $C = 3\mu\text{F}$, and the resistor has resistance $R = 6\text{M}\Omega$.



1. What is the total complex impedance Z ?

First, we need the impedance of each individual component:

	Z
Resistor	R
Capacitor	$\frac{1}{i\omega C} = \frac{-i}{\omega C}$
Inductor	$i\omega L$

Remember, for impedance Z , you add everything up like resistors. We first simplify the inductor and capacitor in parallel, calling this impedance Z_{LC} :

$$\frac{1}{Z_{LC}} = \frac{1}{Z_C} + \frac{1}{Z_L} = i\omega C + \frac{1}{i\omega L}$$

$$Z_{LC} = \frac{1}{i\omega C + \frac{1}{i\omega L}}$$

Multiply the top and the bottom by $i\omega L$ to get rid of the fraction in the denominator:

$$Z_{LC} = \frac{1}{i\omega C + \frac{1}{i\omega L}} \times \left(\frac{i\omega L}{i\omega L} \right)$$

$$= \frac{i\omega L}{1 + (i\omega C)(i\omega L)} = \frac{i\omega L}{1 - \omega^2 LC}$$

Next, add in the resistor in series: this gives

$$Z = Z_R + Z_{LC} = R + \frac{i\omega L}{1 - \omega^2 LC}$$

$$= 6 \times (10^6 + 765.3i)\Omega$$

2. Using the answer from part (a), what is the current $I(t)$ going through the power supply?

We know $V(t)$ looks like $\cos(\omega t)$, so we expect $I(t)$ to look like a cosine with the same frequency ω . The two things we need to figure out exactly what the cosine looks like are the *amplitude* I_0 and the *phase shift* ϕ . The final current will be

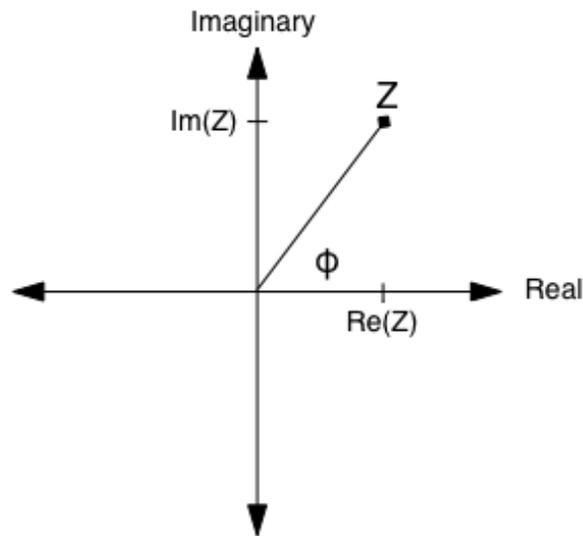
$$I(t) = I_0 \cos(\omega t - \phi)$$

Amplitude

The formula is

$$I_0 = \frac{V_0}{|Z|}$$

So, what is $|Z|$? In short, $|Z| = \sqrt{ZZ^*} = \sqrt{\text{Re}(Z)^2 + \text{Im}(Z)^2}$. What does this mean? The second formula looks a bit like the Pythagorean theorem. This comes from the following: any complex number $Z = X + iY$ can be drawn on the "complex plane," just by plotting the real part (X) on the x -axis and the imaginary part (Y) on the y -axis. See figure 2.



$|Z|$ is just the distance from Z to the origin in this picture, kind of like the absolute value of a real number $|x|$ is the distance from x to the origin. The distance is just the hypotenuse of a right triangle, with $\text{Re}(Z)$ as one leg and $\text{Im}(Z)$ as the other leg.

In our case,

$$|Z| = \sqrt{R^2 + \left(\frac{\omega L}{1 - \omega^2 LC}\right)^2}$$

Then, the magnitude of current I_0 is:

$$I_0 = \frac{V_0}{|Z|}$$
$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\frac{\omega L}{1 - \omega^2 LC}\right)^2}} = 2\text{mA}$$

Phase shift

The phase shift is the angle ϕ that is drawn in figure 2. So, we must have

$$\begin{aligned}\tan \phi &= \frac{\text{Im}(Z)}{\text{Re}(Z)} \\ &= \frac{\omega L}{1 - \omega^2 LC} \frac{1}{R} = \frac{\omega L}{(1 - \omega^2 LC)R} \\ \phi &= \tan^{-1}\left(\frac{\omega L}{(1 - \omega^2 LC)R}\right) = \left(\frac{\omega L}{(1 - \omega^2 LC)R}\right) \\ &= 1.3 \times 10^{-4} \text{rad}\end{aligned}$$

Putting everything together,

$$\begin{aligned}I(t) &= \frac{V_0}{\sqrt{R^2 + \left(\frac{\omega L}{1 - \omega^2 LC}\right)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{(1 - \omega^2 LC)R}\right)\right) \\ &= 2\text{mA} \times \cos(\omega t - 1.3 \times 10^{-4})\end{aligned}$$

3. What is I_{rms} ? What is V_{rms} ?

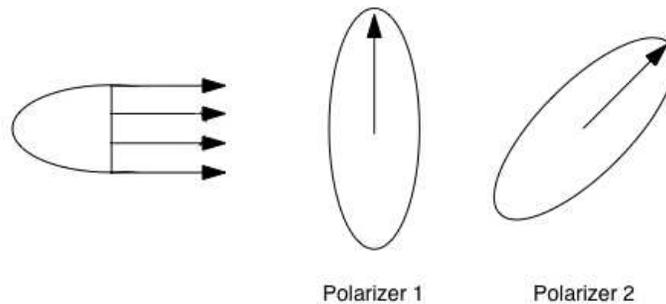
In general, $I_{rms} = \frac{I_0}{\sqrt{2}}$ and $V_{rms} = \frac{V_0}{\sqrt{2}}$, so $V_{rms} = \frac{12\text{V}}{\sqrt{2}} = 8.5\text{V}$ and $I_{rms} = 1.4\text{mA}$.

4. What is the phase difference between $V(t)$ and $I(t)$?

The phase difference is just the number ϕ we found, which is $\phi = 1.3 \times 10^{-4}$ radians. Note that it is very important to use radians!

Problem 2: Polarization

We start with a lamp shining unpolarized light in the \hat{z} direction, with intensity $1.5 \frac{\text{W}}{\text{m}^2}$. The light passes through two polarizers, as shown in the diagram. The first polarizer is oriented in the \hat{x} direction, and the second polarizer is oriented at a 45° angle, pointing in the $\frac{\hat{x} + \hat{y}}{\sqrt{2}}$ direction.



1. After the light passes through the first polarizer, how much intensity I_1 is transmitted?

The quick answer is $\frac{1}{2}I_0$ and $I_1 = 0.75 \frac{\text{W}}{\text{m}^2}$; intuitively, if the light is initially unpolarized, half of the intensity/energy should be in one polarization, and half in the other.

We can get to this answer using the rule

$$I_{trans} = I_0 \cos^2 \theta$$

where θ is the angle between the light polarization and the polarization filter's axis. With unpolarized light, there is no angle θ that we can use; however, we can visualize (in our heads) considering only the little piece of the unpolarized light that points in the small segment $(\theta, \theta + d\theta)$. The average intensity pointed in any particular direction is $\frac{I_0}{2\pi}$, so the amount of intensity between θ and $\theta + d\theta$ is $I_0 \frac{d\theta}{2\pi}$. The amount transmitted is then $I_0 \cos^2 \theta \frac{d\theta}{2\pi}$. Now add up all the little slices, i.e. integrate over θ :

$$\begin{aligned} I_1 &= \int_0^{2\pi} I_0 \cos^2 \theta \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} I_0 \frac{1 + \cos 2\theta}{2} \frac{d\theta}{2\pi} \\ &= \frac{2\pi I_0}{4\pi} = \frac{I_0}{2} = 0.75 \frac{\text{W}}{\text{m}^2} \end{aligned}$$

2. The intensity from (a) gives you the average magnitude of the *Poynting vector* \vec{S}_1 after the first polarizer: $I_1 = \text{average}(|\vec{S}_1|)$. Note that the average value I_1 is exactly $\frac{1}{2}$ of the peak value, $|\vec{S}_1|_{\text{peak}}$. Remember, the Poynting vector tells you the intensity of your light and the direction the light is traveling in. You also know the polarization of the light. So, what is the *peak* magnitude and direction of \vec{E}_1 , the transmitted electric field?

To avoid confusion, we'll work only with the peak values to start with. First, we need the expression for the Poynting vector:

$$\begin{aligned} \vec{S}_1 &= \frac{\vec{E}_1 \times \vec{B}_1}{\mu_0} \\ |\vec{S}_1| &= \frac{|\vec{E}_1| |\vec{B}_1| \sin \theta}{\mu_0} \end{aligned}$$

Since $\theta = 90^\circ$, the sine just becomes 1. We also need to remember that for electromagnetic radiation,

$$|\vec{E}| = \frac{|\vec{B}|}{c}$$

One way to remember this is to think, "for light, the magnetic field is much weaker than the electric field." The electric field is much bigger and more important.

Plugging this in,

$$\begin{aligned} |\vec{S}_1| &= \frac{|\vec{E}_1|^2}{c\mu_0} \\ |\vec{E}_1| &= \sqrt{c\mu_0 |\vec{S}_1|} \end{aligned}$$

Now remember that $|\vec{S}_1|$ is the *peak* intensity; in (a) we solved for the *average* intensity: $I_1 = \frac{1}{2} |\vec{S}_1|$. So,

$$|\vec{E}_1| = \sqrt{c\mu_0 2I_1} = 23.8 \frac{\text{V}}{\text{m}}$$

The polarization of the light is the same direction as the polarization filter, in the \hat{x} direction.

3. After the light hits the second polarizer, what is the final intensity of the light? What is the magnitude and direction of the electric field, \vec{E}_2 ?

We can use the rule $I_{\text{trans}} = I_0 \cos^2 \theta$. Since the polarizers are offset by $\theta = 45^\circ$,

$$\begin{aligned} I_2 &= I_1 \cos^2 45^\circ \\ &= I_1 \frac{1}{2} \\ &= 0.375 \frac{\text{W}}{\text{m}^2} \end{aligned}$$

Using the same reasoning as in (b), the peak electric field is

$$|\vec{E}_2| = \sqrt{c\mu_0 2I_2} = 16.8 \frac{\text{W}}{\text{m}^2}$$

The direction is again the same as the direction of the polarizer, $\frac{\hat{x} + \hat{y}}{\sqrt{2}}$.

Problem 3: Single lenses

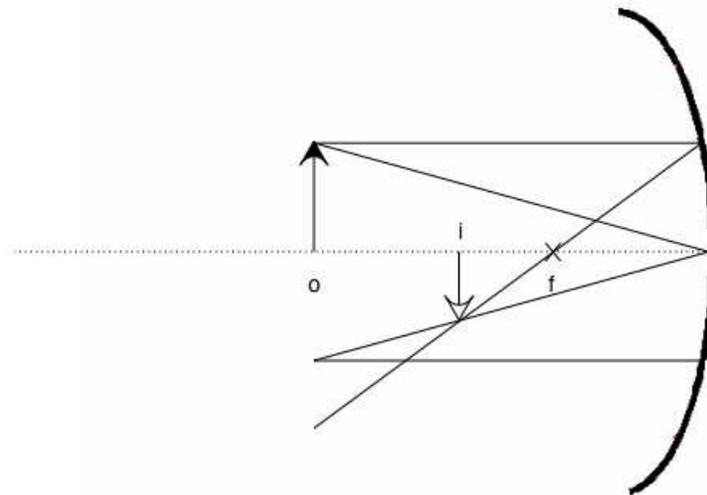
$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f}$$

1. A warmup problem. Suppose you have a concave mirror, with radius $r = 20\text{cm}$. What is the focal length f of the mirror? How would this be different if the mirror were convex? Now suppose you have an object sitting $o = 25\text{cm}$ away from the mirror. Where is the image, i.e. what is i ?

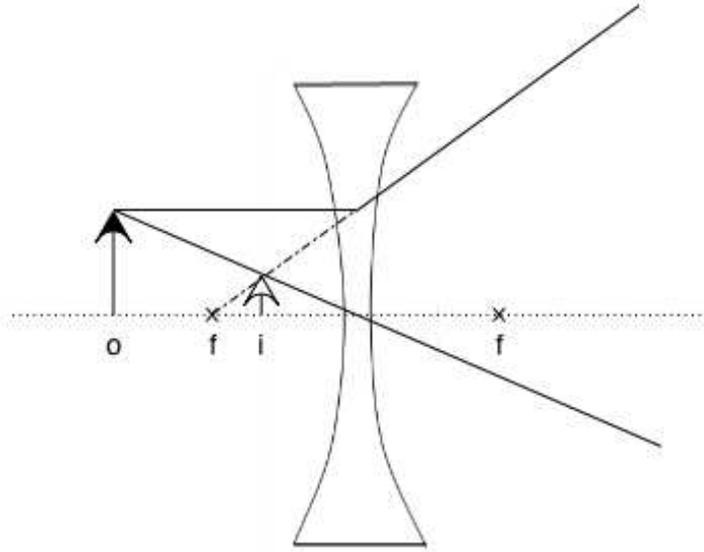
The focal length of a mirror is one-half the radius of the mirror, so $f = \frac{r}{2} = 10\text{cm}$.

The key equation for optics is $\frac{1}{o} + \frac{1}{i} = \frac{1}{f}$. So, $\frac{1}{i} = \frac{1}{10} - \frac{1}{25} = \frac{3}{50}$, and $i = \frac{50}{3} = 16.67\text{cm}$.

2. Ray tracing: draw the appropriate ray tracing diagram for part (a).



3. Diverging lenses: suppose you have a diverging lens with $f = -25\text{cm}$. Place an object of height $h_o = 5\text{cm}$, 40cm away from the lens. First, draw the ray tracing diagram. Then, find the image distance, the image height, and say whether the image is real or virtual.



$$\frac{1}{i} = \frac{1}{f} - \frac{1}{o} = \frac{1}{-25} - \frac{1}{40} = \frac{-13}{200}$$

$$i = -15.4\text{cm}$$

This is a virtual image, since $i < 0$. The height of the image is:

$$M = \frac{h_i}{h_o} = -\frac{i}{o}$$

$$h_i = 5 \frac{15.4}{40} = 1.9\text{cm}$$

The image is virtual, upright, and smaller.

4. Converging lenses: suppose you have a converging lens with $f = 15\text{cm}$. Place an object of height $h_o = 3\text{cm}$ at $o = 10\text{cm}$. Where is the image? How large is the image? Now place the object at $o = 30\text{cm}$. Where is the image? How large is the image?

If $o = 10\text{cm}$, then

$$\begin{aligned}\frac{1}{i} &= \frac{1}{15} - \frac{1}{10} = -\frac{1}{30} \\ i &= -30\text{cm}\end{aligned}$$

This is a virtual image. The image will be magnified: $h_i = h_o \times \frac{-i}{o} = 9\text{cm}$. The image is virtual and upright.

If $o = 30\text{cm}$, then

$$\begin{aligned}\frac{1}{i} &= \frac{1}{f} - \frac{1}{o} = \frac{1}{15} - \frac{1}{30} = \frac{1}{30} \\ i &= 30\text{cm}\end{aligned}$$

Since $i = o$, the magnification factor is $M = \frac{-i}{o} = -1$, so $h_i = -h_o = -3\text{cm}$. The image is real and inverted.

5. A plano-convex lens with radius $r = 29\text{cm}$ is made from glass with a refractive index dependent on the wavelength of light. Suppose a violet lamp ($\lambda \approx 420\text{nm}$) placed at $o = 40\text{cm}$ creates an image at $i = 17.85\text{cm}$, and a red lamp ($\lambda \approx 680\text{nm}$) placed at $o = 40\text{cm}$ creates an image at $i = 15.9\text{cm}$. What is the range of the index of refraction n of the lens? (i.e. what is $n_{red} - n_{violet}$?)

We're given object/image distances for violet and red light, so this lets us calculate the focal lengths, f_{violet} and f_{red} for the lens, for the given wavelengths of light.

$$\begin{aligned}\frac{1}{f_{violet}} &= \frac{1}{o_{violet}} + \frac{1}{i_{violet}} = \frac{1}{40} + \frac{1}{17.85} \\ f_{violet} &= 12.34\text{cm} \\ \frac{1}{f_{red}} &= \frac{1}{40} + \frac{1}{15.9} \\ f_{red} &= 11.38\text{cm}\end{aligned}$$

Now we have to turn to the lensmaker's formula,

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

R_1 is the radius of the first side of the lens (that the light hits), and R_2 is the radius of the second side of the lens (the back side of the lens). The sign convention is called "Cartesian" and works like this: draw the "radius" for side of the lens that you're considering. If you drew a line to the right, i.e. to the "positive x -axis", then R is positive; if you drew a line to the left, i.e. to the "negative x -axis", then R is negative.

We want to solve for n . Suppose the light hits the curved side of the mirror first (it doesn't matter in the end). Draw the radius of this side of the lens; your line should go to the right, i.e. to the "positive x -axis", so R_1 is positive. So $R_1 = +29\text{cm}$. The second side of the lens is flat, so $R_2 = \infty$. Thus:

$$\frac{1}{f} = (n - 1) \left(\frac{1}{29} - \frac{1}{\infty} \right) = \frac{n - 1}{29\text{cm}}$$

So, solving for n ,

$$n - 1 = \frac{29\text{cm}}{f} n = 1 + \frac{29\text{cm}}{f}$$

We want $n_{red} - n_{violet}$, so

$$\begin{aligned}n_{red} - n_{violet} &= \left(1 + \frac{29}{f_{red}}\right) - \left(1 + \frac{29}{f_{violet}}\right) \\ &= \frac{29}{f_{red}} - \frac{29}{f_{violet}} \\ &= 0.20\end{aligned}$$

6. For fun: why won't this plano-convex lens work to focus x-rays?

In general, the index of refraction n depends on the wavelength of light being used. We just found a small correction (0.20) in the previous calculation (note: these numbers were completely made up!), between violet and red light. But x-rays have a much smaller wavelength, factors of 10 smaller than visible light. It turns out $n_{x-ray} \approx 1$ for almost any material you'd want to make a lens from!

So, x-rays are hard to bend and focus. An x-ray machine for your teeth or bones has to use a different method to image a specific part of your body. These machines accelerate electrons into a plate of metal, which creates a spray of x-rays. The x-rays go through a plate of metal with a small hole in it; this ensures that only x-rays going in the proper direction leave the machine (this is something like a pinhole camera).

GOOD LUCK