Notes for Special Relativity, Quantum Mechanics, and Nuclear Physics

1. More on special relativity

Normally, when two objects are moving with velocity v and u with respect to the stationary observer, the relative velocity of one object to the other is just $v + u$. But when the objects' velocities are comparable to the speed of light, special relativity effect is more noticeable, and you need the following velocity transformation (to keep the speed of light constant):

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

Where u and v are measured with respect to the ground, u' is the velocity of object 1 relative to object 2 (moving in the same direction). Notice that if both u and v are very small compared to c, then $u' = u - v$ like normal.

If you're given the relative velocity and velocity of one object with respect to the ground, and asked to find the other velocity with respect to the ground, then use:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Relativistic momentum:

$$p = \frac{mv}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \gamma mv$$

Relativistic total energy:

$$E = \gamma mc^2 = \sqrt{p^2c^2 + (mc^2)^2} = c\sqrt{p^2 + m^2c^2}$$

Rest energy (energy stored as mass):

$$E_{rest} = mc^2$$

2. Quantum mechanics

Photon energy: $E_\gamma = hf$

Where γ denotes the photon (like gamma ray), $f = c/\lambda$ is the frequency, and h is the Planck's constant:

$$h = 4.14 \times 10^{-15} eVs = 6.63 \times 10^{-34}\ Js$$

(note: $1eV = 1.6 \times 10^{-19} J$)

<u>Photoelectric effect</u>:

Here's a link you can play with:

http://phet.colorado.edu/simulations/sims.php?sim=Photoelectric_Effect

(It has other effects too.)

The basic idea is photons hit the metal plate and bump electrons out of one plate to the other, creating a current. By connecting a battery to the plates, you apply a voltage that can stop the electron flow. Each kind of metal has a specific minimum voltage that can stop the electrons from leaving the metal, and it's called the "work function" of the metal. So each work function is a constant and can be understood as the minimum kinetic energy an electron must receive from a photon to overcome its potential energy in the atom. The kinetic energy of the electron is:

$$K = hf - \phi$$

Where ϕ is the work function.

Example: Problem 15 (chapter 34)
When the metal plate has light of wavelength λ shone on it, the electrons have kinetic energy 2.8eV. When the wavelength increases by 50%, the electrons' kinetic energy is 1.1eV. Find the original wavelength and the work function of the metal.

Solution:

$$\lambda = 150\% \times \lambda_0 = \frac{3}{2}\lambda_0 \qquad \Longrightarrow \qquad f = \frac{2}{3}f_0$$

$$K_0 = hf_0 - \phi = 2.8eV$$

$$K = hf - \phi = \frac{2}{3}hf_0 - \phi = 1.1eV$$

Subtract the two kinetic energies from each other:

$$K_0 - K = hf_0 - \phi - \left(\frac{2}{3}hf_0 - \phi\right) = \frac{1}{3}hf_0 = (2.8 - 1.1)eV = 1.7\ eV$$

$$\Longrightarrow f_0 = \frac{3(1.7eV)}{4.14 * 10^{-15}eVs} = 1.23 * 10^{15}\ Hz$$

$$\Longrightarrow \lambda_0 = \frac{c}{f_0} = \frac{3 * \frac{10^8 m}{s}}{1.23 * \frac{10^{15}}{s}} = 2.44 * 10^{-7}m = \mathbf{244nm}$$

$$\phi = hf_0 - K_0 = 3(1.7eV) - 2.8eV = \mathbf{2.3eV}$$

<u>Bohr Model</u>:
This is a simple model of electrons orbiting the nucleus in circular orbits. The electrons experience only Coulomb force from the nucleus which keeps them in orbit:

$$F = \frac{kZe^2}{r^2}$$

(where Z is the atomic number, i.e. the number of protons in the nucleus). So the potential energy of the electron is:

$$E = -\frac{kZe^2}{r}$$

Add the kinetic energy (which is half the potential energy – a derivation can be found on Wikipedia) to get the total energy:

$$E = -\frac{kZe^2}{2r}$$

The radius of orbit depends on the so-called quantum number n:

$$r_n = \frac{n^2\hbar^2}{Zke^2m_e}, \qquad n = 1,2,3\ldots$$

When $n = 1$, for H atom (Z=1), we have the smallest radius (the Bohr radius):

$$r_1 = a_0 = \frac{\hbar^2}{ke^2m_e} = 0.53 * 10^{-10}\ m$$

The energy of the electron at the n-th orbit is:

$$E_n = -\frac{kZe^2}{2r_n} = -\frac{kZe^2}{2}\times\frac{Zke^2m_e}{n^2\hbar^2} = -\frac{Z^2ke^2}{2a_0n^2}$$
$$= -Z^2\times\frac{(ke^2)^2m_e}{2\hbar^2n^2} = \boldsymbol{-13.6eV\times\frac{Z^2}{n^2}}$$

Note: $\hbar = \frac{h}{2\pi} = 1.05\times10^{-34}Js = 6.58\times10^{-16}\ eVs$

This is just all equation acrobatics. I suggest that you go through the equations yourself to see how they are related. The gist is **electron energy in Bohr model is proportional to $\boldsymbol{1/n^2}$** (n=1, 2, 3, …).

When the electron changes from orbit n_1 to orbit n_2, it either absorbs or emit a photon.
If $n_1 < n_2$, i.e. the electron goes farther away from the nucleus, it gains potential energy (think about gravitational potential) and absorbs a photon.
If $n_1 > n_2$, i.e. the electron gets closer to the nucleus and loses potential energy, it emits a photon.
The energy of the photon is:

$$E_\gamma = 13.6eV\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \quad (= hf)$$

<u>De Broglie wavelength</u>

Wavelength of any object with mass and velocity:

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

The de Broglie wavelength (or actually the wave-particle duality) helps explaining why the energy levels in hydrogen atom (Bohr model) are proportional to $1/n^2$. See this page if you're interested: http://www.colorado.edu/physics/2000/quantumzone/debroglie.html

Quantum mechanical effect is only noticeable when the wavelength is large enough. That's why we don't observe ourselves and things in our daily life behave like wave. However, the larger the wavelength is, i.e. the wider the "wave packet" is, the less certain we can locate a particle inside its wave packet (it can be anywhere in that wave packet). Hence Heisenberg formulates the Uncertainty Principle:

$$\Delta x\,\Delta p \geq \hbar$$

Because

$$\Delta E = \frac{1}{2}mv^2 = \frac{1}{2}pv \approx \Delta p\, v$$

(we ignore factor of 2 because this is just an approximation).
And $\Delta x = v\Delta t$, we have:

$$\Delta x\Delta p = v\Delta t\,\Delta p = \boldsymbol{\Delta E\,\Delta t} \geq \hbar$$

A point worth noting:
Since particles can behave like wave, you can picture them as springs which oscillate with frequency ω. The energy level for a spring is:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Where n = 0, 1, 2… When n=0, you have the ground state energy $E_0 = \frac{\hbar\omega}{2}$. But this is **not** the ground state energy of an electron orbiting a nucleus in the Bohr model.

Since each particle can be described (and they also behave) as a wave, when we want to study the wave's behavior, we use the "wave function" ψ (psi), which depends on position x.

Recall that for a particle, the total energy is the sum of kinetic and potential energy:

$$E = K + U = \frac{1}{2}mv^2 + U = \frac{1}{2m}(mv)^2 + U = \frac{p^2}{2m} + U$$

Well, in quantum mechanics, the momentum is redefined as a derivative acting on the wave function:

$$p^2 = -\hbar^2 \frac{d^2}{dx^2}$$

i.e.

$$p^2\psi = -\hbar^2 \frac{d^2\psi}{dx^2} = -\hbar^2\psi''$$

So now we multiply the energy equation above with ψ and get the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\psi'' + U\psi = E\psi$$

$$\Rightarrow \psi'' = -\frac{2m}{\hbar^2}(E - U)\psi = -k^2\psi \quad \left(k = \frac{2m}{\hbar^2}(E - U)\right)$$

A solution to this equation has the form:

$$\psi = A\sin kx$$

Because if we take the second derivative of ψ we retrieve the **Schrodinger equation**:

$$\psi'' = \frac{d}{dx}(kA\cos kx) = -k^2 A\sin kx = -k^2\psi$$

Now that we have the general form of the equation, we must find the constants A and k specific to each problem. The most classic problem of quantum mechanics is the infinite square well.

A **one-dimensional infinite square well** with size L has potential energy set up as follows:

$$U = \begin{cases} 0 & 0 < x < L \\ \infty & x < 0 \text{ and } x > L \end{cases}$$

(If it is 2-dimensional, with infinitely high walls on all 4 sides, it's really a "well" that we normally think of, and U and ψ would depend on both x and y. But because it's just 1D, the potential and the wave function only depend on x.)

Since the potential energy is infinite outside the well, the particle has no chance of being outside the well – it can't climb the infinitely high wall to get to the outside. Another way of looking at it is if U is infinitely large, then K must be 0, because the total energy cannot be greater than infinity and kinetic energy cannot be negative. Hence, no particle movement, i.e. no wave outside the well.

Inside the well, U =0, substitute this into the Schrodinger equation and we have:

$$-\frac{\hbar^2}{2m}\psi'' = E\psi, \qquad k^2 = 2mE/\hbar^2$$

We also know that the wave function is a sine function, and that it has to be 0 at the endpoints:

$$\psi(x=0) = \psi(x=L) = 0$$

to match with the condition that no wave exists outside the well (Picture a string held at both ends, the ends of the string are fixed and the wave function is 0 there, no matter how each point along the string vibrates.) So:

$$\sin(k*0) = \sin(kL) = 0$$

The only way for $\sin kL = 0$ is to have

$$\boldsymbol{k} = \frac{\boldsymbol{n\pi}}{\boldsymbol{L}}, \qquad n = 1, 2, 3, \ldots$$

Such that $\sin kL = \sin n\pi = 0$

Putting this expression and the energy together, we have the energy levels for a particle in a 1D infinite square well:

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

Now that we've determined the constant k in $\psi = A\sin(kx)$, we need to determine the constant A to have a complete wave function specific to the 1D infinite square well problem.

Let us go back to the uncertainty principle: a particle can be anywhere inside its wave packet at a given time, that means each point inside the wave packet has a probability of the particle being there. Mathematically, the probability is defined based on the wave function:

$$P(x) = \int \psi^2 dx$$

The limits of integration are the limits of the region you want to look for the particle. So the total probability of the particle in the whole square well must be 1.

$$1 = \int_0^L \psi^2 dx = \int_0^L A^2 (\sin \mathrm{kx})^2 \mathrm{dx}$$

$$= A^2 \int_0^L \sin^2(kx)\ dx = A^2 \left(\frac{x}{2} - \frac{\sin 2kx}{4k}\right)_0^L$$

$$= A^2 \left(\frac{L}{2}\right)$$

$$\Rightarrow \boldsymbol{A} = \sqrt{\frac{\boldsymbol{2}}{\boldsymbol{L}}}$$

So the final wave function of a particle in 1D infinite square well of size L is:

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$$

Example problems:

#18: Find the width of an infinite square well in which a proton cannot have energy less than 100eV.

Solution: Since $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \geq 100eV$, and we see that as n increases, E increases, so we know the minimum energy occurs when n=1, and:

$$100eV = \frac{\pi^2\hbar^2}{2mL^2}$$

Then $L = \sqrt{\frac{2m_p}{\pi^2\hbar^2} * 100eV}$, where $m_p = 938MeV$

#44: A particle is trapped inside an infinite square well of side L. Assuming it's in its ground state, find the probability of finding the particle in a region of 80% the well's width and centered around the point L/2

Solution:
The particle is in its ground state, so n=1.
The region is 80% of the well, centered around the midpoint, so it is 40% of L on each side, therefore the limits of integration are 0.1L and 0.9L.

$$P = \int \psi^2 dx = \frac{2}{L}\int_{0.1L}^{0.9L} \left(\sin\frac{\pi x}{L}\right)^2 dx = 0.987 = 98.7\%$$

Always check that $P \leq 1$.

3. Radioactive decay

For a sample with N_0 atoms originally, after some time t it will decay and have only $N < N_0$ atoms. Suppose the decay rate is proportional to the number of atoms present, i.e.:

$$\frac{dN}{dt} = -\lambda N$$

Where λ is called the decay constant. Then we can integrate to find the relation between N and N_0:

$$\int_{N_0}^{N} \frac{dN}{N} = \int_0^t -\lambda dt \quad \Rightarrow \quad \boldsymbol{N = N_0 e^{-\lambda t}}$$

Notice that the exponent is negative, so the number of atoms only decreases as time progresses. And when $t = 1/\lambda$, $N = N_0/e$, i.e. the number of atoms has become exactly e times less than the original quantity. But e is a funny number and we don't want to deal with it, so we change this formula to something more useful: how long does it take for the sample to reduce its size by half?

$$\frac{N}{N_0} = \frac{1}{2} = e^{-\lambda t_{1/2}}$$

Where $t_{1/2}$ is called the "half life" of an element. Take the natural log of both sides and solve for the half life:

$$\ln\left(\frac{1}{2}\right) = -\lambda t_{1/2} = -\ln 2 \quad \Rightarrow t_{1/2} = \frac{\ln 2}{\lambda}$$

And meanwhile:

$$N = N_0 \exp(-\frac{\ln 2}{t_{1/2}} \times t)$$

$$\Rightarrow \boldsymbol{N = N_0 \times 2^{-t/t_{1/2}}}$$

The decay rate is called the "activity", with unit Becquerel (Bq), where 1Bq = 1 decay/s.

Example problem:
(a) A sample has activity 2.4MBq at the beginning. After 30 minutes its activity drops to 1.9MBq. Find the half life of the element.
(b) Suppose the sample is moving with speed v = 0.8c. Find the time it takes for the activity to drop to one fourth its original value.

Solution:
(a)

$$1.9\text{MBq} = 2.4\text{MBq} \times 2^{\frac{-30\text{ minutes}}{\text{t}_{\text{half}}}}$$

$$\Longrightarrow \ln\left(\frac{1.9}{2.4}\right) = -\frac{30}{t_{half}}\ln 2 \quad \Longrightarrow \; t_{half} = -\frac{30\ln 2}{\ln\left(\frac{1.9}{2.4}\right)} = 89\; minutes$$

(b) To have $N = \frac{1}{4} N_0$, the sample must go through 2 half lives. But since it's moving relativistically fast, its half life is lengthened (time dilation):

$$t_{half} = \gamma t_{half\,0} = t_{half\,0}\frac{1}{\sqrt{1-0.8^2}} = \frac{5}{3}t_{half\,0}$$

So the time in question is:

$$t = 2t_{half} = \frac{10}{3}\; t_{half\,0} = \frac{10}{3}\;(89\; minutes) \approx 297\; minutes$$