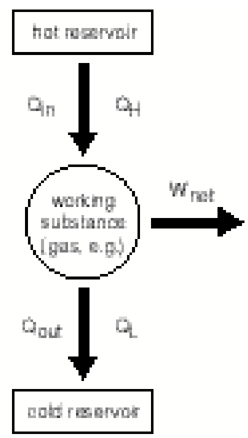
T-5. Engines and Efficiency

Solutions to Dicussion Questions

1. In common-sense language, efficiency is $\frac{\text{"what you want"}}{\text{"what you have to put in"}}$. Let's see how this common-sense notion of efficiency applies to heat engines and refrigerators.

a) A heat engine, such as a steam engine, takes advantage of the everyday fact that "heat wants to flow from hot to cold." The engine "siphons off" some of this flowing heat energy in the form of useful work. This is shown in the schematic diagram at right.

Keeping in mind the common-sense meaning of efficiency, how would you define the efficiency of a heat engine? Your definition of $e_{\text{heat engine}}$ should involve the quantities W_{net}, Q_H, and/or Q_L.



Schematic diagram of a heat engine, showing energy flow.

For a heat engine, *what you have to put in* is heat energy. *What you get out* is useful work. So the efficiency of a heat engine should be (and is) defined as

$$e_{\text{heat engine}} = W_{net}/Q_{in}.$$

b) Use the First Law of Thermodynamics, together with the fact that the engine runs through a *cycle*, to show that your definition is equivalent to $e_{\text{heat engine}} = 1 - Q_L/Q_H$.

The First Law requires that over a complete cycle of operation, the net heat added, the net work done, and the net change in the internal energy of the working substance will satisfy

$$\Delta E_{int} = W_{net}^{\text{on gas}} + Q_{net}^{\text{added}}.$$

But over the course of a cycle, the gas returns to its original state, so there can be no net change in the internal energy. Thus $\Delta E_{int}^{\text{cycle}} = 0$, so we have

$$0 = W_{net}^{\text{on gas}} + Q_{net}^{\text{added}}.$$

But the net heat added Q_{net}^{added} is just $Q_{in} - Q_{out}$. And the net work done *on the gas* is minus the net work done *by the gas*, which is W_{net} in the diagram: $W_{net}^{\text{on gas}} = -W_{net}$. So we have

$$0 = -W_{net} + Q_{in} - Q_{out}$$

or

$$W_{net} = Q_{in} - Q_{out}.$$

All these minus signs are confusing, but if you take a minute to see where we've ended up, you'll see that this last equation makes a lot of sense: The net work output of a heat engine is equal to the heat energy you put in, less the heat that gets dumped out by the engine. This is really just conservation of energy (which is what the First Law is all about).

Now we can re-express the efficiency as

$$e_{heat\ engine} = (Q_{in}-Q_{out}) / Q_{in} = 1 - Q_{out}/Q_{in} = 1 - Q_L/Q_H$$

as desired.

c) Your refrigerator forces heat energy to flow "against the grain," from the cold icebox to the warm kitchen. Naturally this requires an input of work, as shown in the schematic diagram (below).

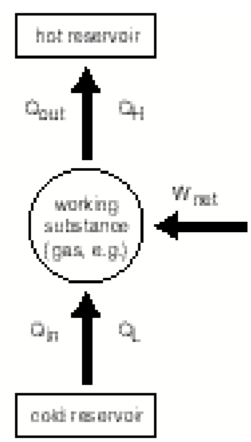
Keeping in mind the common-sense definition of efficiency, how would you define the efficiency of a refrigerator? We actually call this the *coefficient of performance*, $K_{refrigerator}$. Your definition of $K_{refrigerator}$ should involve the quantities W_{net}, Q_H, and/or Q_L.

What you want is to remove heat from the inside of your refrigerator (the cold reservoir). *What you have to put in* is work. (The electricity from your wall socket drives a motor that does this work).

So the "efficiency" of a refrigerator should be defined as

$$K_{ref} = Q_{in}/W_{net} = Q_L/W_{net}.$$

This is the right answer, but since Q_L/W is not a number between 0 and 1, we call this ratio something other than "efficiency": the coefficient of performance.



Schematic diagram of a refrigerator, showing energy flow.

d) In the winter, a heat pump uses energy to extract heat from the cold outdoors and pump it into your warm house. (So a heat pump is like a refrigerator for the outside air!) How would you define the efficiency, or coefficient of performance, for a heat pump? Your definition of K_{hp} should involve the quantities W_{net}, Q_H, and/or Q_L.

What you want is to add heat to the air inside your house (the hot reservoir). *What you have to put in* is work. (The electricity from your wall socket drives a motor that does this work).

So the “efficiency” of a heat pump should be defined as

$$K_{HP} = Q_{out}/W_{net} = Q_H/W_{net}.$$

This is the right answer, but since Q_L/W is not a number between 0 and 1, we call this ratio something other than “efficiency”: the coefficient of performance.

When we say that a heat pump is like a refrigerator for the outside air, this is literally true. You could actually make a crude heat pump by setting a refrigerator in the window, facing outwards, with the doors open. The “waste heat” from the back of the refrigerator then heats the house.

2. What does it mean for an engine to operate on the Carnot cycle?

It means that the engine contains some compressible substance, a gas maybe, and it runs this gas through a cycle of four reversible steps:

First, the gas is at some high temperature T_H, and it is allowed to expand in such a way that its temperature remains constant. (This might be done by immersing everything in a "heat bath" at temperature T_H, which is so large that it can provide the necessary heat inflow without its temperature changing.)

Second, the gas is allowed to expand again, but this time in such a way that no heat flows out. (To do this we would have to insulate the container by removing it from the heat bath.) This causes the temperature of the gas to drop to some cooler temperature T_C.

Third, the gas is compressed in such a way that its temperature remains constant. (For this we need a second heat bath to absorb the heat outflow, this one at the lower temperature T_C.)

Fourth, the gas is compressed further, this time in such a way that no heat flows out. (So again we insulate the container by removing it from the heat bath.) This causes the temperature of the gas to increase, and in fact, we set things up so that the gas returns to the original starting temperature T_H. Now we are ready to repeat the cycle.

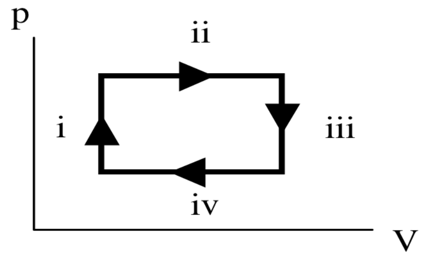
3. Does the working substance of a Carnot engine have to be an ideal gas? If a Carnot engine uses a different substance, then can we still find the efficiency using the formula $e_C = 1 - T_L/T_H$?

Nothing in the Carnot cycle requires that the gas be *ideal*. For example, the working substance could be a van der Waals gas (which is a more realistic kind of gas in which the particles interact weakly with one another). Or it could be steam. All that matters is that the steps be as follows: isothermal expansion at T_H, adiabatic expansion to T_C, isothermal compression at T_C, adiabatic compression to T_H. And as long as these steps are done reversibly, the efficiency will be given by the usual formula.

Admittedly, if we use a van der Waals gas or something, then our equation of state becomes more complicated than $pV = NkT$. Correspondingly, it might become difficult to draw the transformations on the p-V diagram---for example, isotherms won't be hyperbolas anymore. Thus, when we sketch a Carnot cycle on the p-V diagram using the usual curves pV = const. and pV^γ = const. for the transformations, then we are actually assuming that an ideal gas is being used.

It will turn out that the result for the *efficiency* of a Carnot engine operating between two temperatures will always be the same. This result is proved in the second part of the suplimentary worksheet TS-2.

4. A cyclic heat engine uses an ideal gas for its working substance. The engine operates on the following four-step process.



a) Think about what is going on during the first step of the cycle. Then decide: Is heat flowing into the gas or out of the gas during this step? (Think about the First Law. You should not need to calculate anything in order to decide.)

> Notice that the volume is remaining constant, but the pressure is increasing. So from $pV = NkT$, we can see that the temperature of the gas must be increasing. Hence, the internal energy of the gas must be increasing.
>
> If the gas is gaining more and more internal energy, then this energy must be coming from somewhere. So either we must be doing work on the gas, or else heat must be flowing into the gas, or else there is some combination of these. But since the volume of the gas is remaining constant, we cannot be doing any work on the gas. So heat must be flowing in.

b) Is the efficiency of this engine given by $e = 1 - T_L/T_H$? If not, how could you calculate the efficiency?

> No. The cycle on which this engine runs is "isochoric, isobaric, isochoric, isobaric." This is not the Carnot cycle, which is "isothermal, adiabatic, isothermal, adiabatic." And the formula $e = 1 - T_L/T_H$ is only applicable for the Carnot cycle.
>
> To compute the efficiency of the above cycle, you would have to apply the more fundamental definition $e = W_{net} / Q_{in}$. (Note that heat flows into the gas only during legs (i) and (ii). On the other legs, the heat flows out.)

Answers to Problems

1. a) Heat flows *out of* the gas in step AB and heat flows *into* the gas in steps BC and CA.

b)

Step	ΔE_{int}	W	Q
A → B	$-\frac{3}{2}p_0\left(V_A - V_B\right)$	$-p_0\left(V_A - V_B\right)$	$-\frac{5}{2}p_0\left(V_A - V_B\right)$
B → C	$\frac{3}{2}p_0\left(V_A - V_B\right)$	0	$\frac{3}{2}p_0\left(V_A - V_B\right)$
C → A	0	$p_0 V_A \ln\left(\frac{V_A}{V_B}\right)$	$p_0 V_A \ln\left(\frac{V_A}{V_B}\right)$
		$p_0\left(V_A \ln\left(\frac{V_A}{V_B}\right) - \left(V_A - V_B\right)\right)$ ↑ W_{net}	$p_0\left(V_A \ln\left(\frac{V_A}{V_B}\right) + \frac{3}{2}\left(V_A - V_B\right)\right)$ ↑ Q_{in}

c) This is a *heat engine* since there is a net work done by the gas. Also, the cycle goes around in a *clockwise* fashion on the pV diagram.

d) $$e = \frac{W_{net}}{Q_{in}} = \frac{V_A \ln\left(\frac{V_A}{V_B}\right) - \left(V_A - V_B\right)}{V_A \ln\left(\frac{V_A}{V_B}\right) + \frac{3}{2}\left(V_A - V_B\right)}$$

2. a)

Step	ΔE_{net}	W	Q
1 → 2	$6p_0V_0$	$4p_0V_0$	$10p_0V_0$
2 → 3	$-9\left(1-2^{-2/5}\right)p_0V_0$	$9\left(1-2^{-2/5}\right)p_0V_0$	0
3 → 4	$-\frac{3}{2}\left(3\cdot 2^{3/5}-1\right)p_0V_0$	$-\left(3\cdot 2^{3/5}-1\right)p_0V_0$	$-\frac{5}{2}\left(3\cdot 2^{3/5}-1\right)p_0V_0$
4 → 1	$\frac{3}{2}p_0V_0$	0	$\frac{3}{2}p_0V_0$
		$\left(14-15\cdot 2^{-2/5}\right)p_0V_0$ ↑ W_{net}	$\frac{23}{2}p_0V_0$ ↑ Q_{in}

$$e = \frac{W_{net}}{Q_{in}} = \frac{28-15\cdot 2^{3/5}}{23} \approx 0.229$$

b) Highest: Point 2 $T_2 = 6p_0V_0 / Nk$
Lowest: Point 4 $T_4 = p_0V_0 / Nk$

c) $e_C = 1 - \frac{T_{low}}{T_{high}} = \frac{5}{6} \approx 0.833$

d) These answers are consistent - the Carnot efficiency would be significantly more efficient than the engine shown.