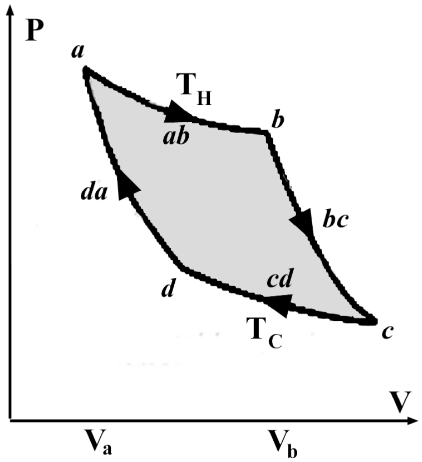
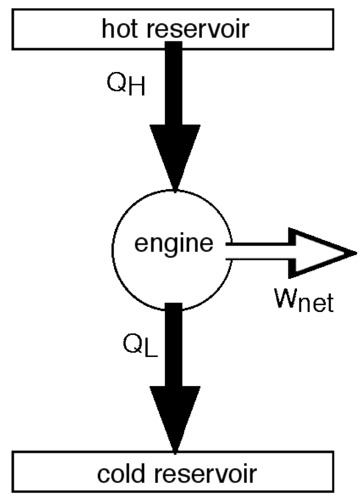
T-S2. Efficiency of the Carnot Engine

Part 1: The Hard Way



The Carnot Engine



i) *For each 'corner' of the path, labeled* ***a****,* ***b****,* ***c****, and* ***d*** *in the PV diagram above, find the missing pressures, volumes, and internal energies in the table below.*

The first and easiest piece of this table to fill in are the internal energies. Assume that the working substace of the engine is an ideal gas with *d* degrees of freedom per particle. Then, the equipartition theorem tells us that:

$$U = \frac{d}{2} NkT$$

The next pieces to fill in are the pressures for points ***a*** and ***b***, which can just be found using the ideal gas law:

$$PV = NkT$$

Now we have to find the volume for point ***c***. ***b*** and ***c*** are connected by an *adiabatic* transformation. The law relating pressures and volumes for an adiabatic process is [see Challenge Problem T-4 for a derivation]

$$P_b V_b^\gamma = P_c V_c^\gamma$$

We can pull out one factor of *V* from both sides and use the ideal gas law to manipulate this equation into one that relates temperature and volume:

$$P_b V_b V_b^{\gamma-1} = P_c V_c V_c^{\gamma-1}$$
$$NkT_H V_b^{\gamma-1} = NkT_L V_c^{\gamma-1}$$

Solving for V_c gives:

$$V_c^{\gamma-1} = \frac{T_H}{T_C} V_b^{\gamma-1}$$

$$V_c = \left(\frac{T_H}{T_C}\right)^{\frac{1}{\gamma-1}} V_b$$

We can make the exponent a little cleaner by recalling that $\gamma = \frac{d+2}{d}$. Therefore, $\frac{1}{\gamma-1} = \frac{d}{2}$. A similar calculation will connect points ***a*** and ***d***. Finally, we use the ideal gas law again to get the remaining pressures.

Point	T	U	V	P
a	T_H	$\frac{d}{2}NkT_H$	V_a	$\frac{NkT_H}{V_a}$
b	T_H	$\frac{d}{2}NkT_H$	V_b	$\frac{NkT_H}{V_b}$
c	T_C	$\frac{d}{2}NkT_C$	$\left(\frac{T_H}{T_C}\right)^{d/2} V_b$	$\left(\frac{T_C}{T_H}\right)^{d/2} \frac{NkT_C}{V_b}$
d	T_C	$\frac{d}{2}NkT_C$	$\left(\frac{T_H}{T_C}\right)^{d/2} V_a$	$\left(\frac{T_C}{T_H}\right)^{d/2} \frac{NkT_C}{V_a}$

ii) *Find the change in internal energy, the work, and the heat associated with each of the four legs of the cycle, labeled ab, bc, cd, and da in the PV diagram above. Fill in your answers in the table below.*

For this part we can use the results from the previous suppliment, TS-1: Ideal Gas Transformations. There, we found that for an isothermal process taking place at temperature T and going from volume V_i to V_f the change in internal energy was 0 (by the equipartition theorem!), the work was $NkT\ln\left(\frac{V_f}{V_0}\right)$, and the heat was $NkT\ln\left(\frac{V_f}{V_0}\right)$ (by the First Law). Plugging in the initial and final states that we found in part (i) gives us the information we need for legs ***ab*** and ***cd*** of the transformation! One thing to note is how we simplified the logarithm part of the expression for path ***cd***:

$$\ln\left(\frac{V_d}{V_c}\right) = \ln\left[\frac{(T_H/T_C)^{d/2} V_a}{(T_H/T_C)^{d/2} V_b}\right] = \ln\left(\frac{V_a}{V_b}\right) = -\ln\left(\frac{V_b}{V_a}\right)$$

Next we recall the results from an adiabatic transformation that started at temperature T_0 and volume V_0 and ended at temperature T_f. The defining property of the adiabatic transformation was that there was no heat transfer (Q = 0). The change in internal energy was simply

$\frac{d}{2}Nk\left(T_f - T_0\right)$ (by the equipartition theorem) and the work was $-\frac{d}{2}Nk\left(T_f - T_0\right)$ (by the first law). Note that these *only* depend on the temperatures and the number of degrees of freedom and not the volumes! Plugging in the initial and final temperatures for paths ***bc*** and ***da*** fills in the rest of the table:

Leg	ΔU	W	Q
ab	0	$NkT_H \ln\left(\frac{V_b}{V_a}\right)$	$NkT_H \ln\left(\frac{V_b}{V_a}\right)$
bc	$\frac{d}{2}Nk\left(T_H - T_C\right)$	$-\frac{d}{2}Nk\left(T_H - T_C\right)$	0
cd	0	$-NkT_C \ln\left(\frac{V_b}{V_a}\right)$	$-NkT_C \ln\left(\frac{V_b}{V_a}\right)$
da	$-\frac{d}{2}Nk\left(T_H - T_C\right)$	$\frac{d}{2}Nk\left(T_H - T_C\right)$	0

iii) *What net work, W_{net}, does one complete cycle of our Carnot engine output?*

The net work is found by adding up the four works we found in part (ii). Note that if we add the two works from the adiabatic processes together we just get 0! The logarithm and *Nk* pre-factors are common for both the isothermal processes, so the net work is merely

$$\boxed{W_{net} = Nk\left(T_H - T_C\right)\ln\left(\frac{V_b}{V_a}\right)}.$$

iv) *Which legs have a positive heat transfer? That is, in which steps do we put heat into the gas? What is the total heat input, Q_{in}, of one complete cycle?*

Only the isothermal legs of the cycle have a non-zero heat transfer. The question of whether or not any given step will have a positive heat transfer (heat going *into* the gas) or not is answered by analyzing the logarithm. Since the volume at point ***b*** is larger than the volume at point ***a***, then the ration $V_b/V_a > 1$, and the log of a number greater than 1 is positive. Therefore, *only leg **ab** has a positive heat transfer*.

$$\boxed{Q_{in} = NkT_H \ln\left(\frac{V_b}{V_a}\right)}.$$

v) *What is the efficiency of this Carnot engine?*

The efficiency is "what we want"/ "what we put in". We 'want' our engine to convert heat into work, so the efficiency will be

$$e = \frac{W_{net}}{Q_{in}}$$

Note that we just used the input heat and not the net heat! Plugging in our results from parts (iii) and (iv) gives

$$e_C = \frac{Nk\left(T_H - T_C\right)\ln\left(\frac{V_b}{V_a}\right)}{NkT_H \ln\left(\frac{V_b}{V_a}\right)} = \frac{T_H - T_C}{T_H}$$

Minor algebra gives us our final answer!

$$\boxed{e_C = 1 - \frac{T_C}{T_H}}.$$

Part 2: The Easy Way

i) *What pieces make up the 'universe' shown?*

The 'universe' for our Carnot engine merely consists of three pieces: The hot reservoir, which will always be at a temperature T_H, the cold reservoir, which will always be at a temperature T_C, and the working substance of the engine [in Part 1 of this worksheet, we used an ideal gas - here the working substance can really be anything as long as we follow the path isotherm-adiabat-isotherm-adiabat to run the engine!]

ii) *What is the total entropy change of the working substance of the engine after one full cycle?*

Entropy is a *state variable*! A key feature of an engine is that it is reusable. After one full cycle, we wind up at the starting point. Since we start and end at the same point in the working substance, the entropy change is simply 0!

$$\boxed{\Delta S_{\text{engine}} = 0}.$$

iii) *What is the entropy change for each of the other elements of the universe found in part (i)?*

We are dealing with all reversible processes, so we can use the entropy formula:

$$\Delta S = \int \frac{dQ}{T}$$

For the reservoirs, we are always at the same temperature, so T can be pulled out of the integral, leaving:

$$\Delta S = \frac{1}{T}\int dQ = \frac{Q}{T}$$

For the hot reservoir, the change in heat, Q, is *negative*, since the reservoir is transfering a heat Q_{in} *into* the engine. Similarly, for the cold reservoir, the change in heat is positive and equal to Q_{out}, since it is accepting all of the exhaust heat from the engine. Therefore,

$$\Delta S_{hot} = -\frac{Q_{in}}{T_H} \text{ and } \Delta S_{cold} = \frac{Q_{out}}{T_C}.$$

iv) *What is the total entropy change of the universe for one cycle?*

To get this, just add together the entropies of the three pieces discussed above:

$$\Delta S_{universe} = \frac{Q_{out}}{T_C} - \frac{Q_{in}}{T_H}.$$

v) *Given that the Carnot engine is reversible, what is the relation between the heats and the temperatures?*

For a reversible process, the total entropy change of the universe is 0. Setting the entropy change of the universe that we got in part (iv) equal to 0 gives the relation between the heats and temperatures:

$$\frac{Q_{out}}{Q_{in}} = \frac{T_C}{T_H}.$$

vi) *Use the definition of efficiency and your result from (v) to find the efficiency of the Carnot engine.*

The efficiency of an engine is $e = \frac{W}{Q_{in}}$. The first law, when applied to an engine, says that the *net* heat input will equal the net work output, so $W = Q_{in} - Q_{out}$, which means we can rewrite the efficiency as $e = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}}$. Using the result from part (v) we can rewrite this in terms of the temperatures and get the general result for the efficiency of any Carnot Engine:

$$e_C = 1 - \frac{T_C}{T_H}.$$