232B Lecture Notes
Abelian Higgs Model

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1 The Model

The abelian Higgs model is a model with U(1) gauge group that is spontaneously broken. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V, \tag{1.1}$$

where

$$V = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2. \tag{1.2}$$

The covariant derivative is

$$D_\mu = \partial_\mu - ieA_\mu. \tag{1.3}$$

It is invariant under the gauge transformation

$$\phi \to \phi' = e^{ie\theta}\phi, \qquad A_\mu \to A'_\mu = A_\mu + \partial_\mu\theta. \tag{1.4}$$

Let us verify

$$D'_\mu\phi' = (\partial_\mu - ieA'_\mu)\phi' = (\partial_\mu - ieA_\mu - ie\partial_\mu\theta)e^{ie\theta}\phi = e^{ie\theta}(\partial_\mu - ieA_\mu)\phi = e^{ie\theta}D_\mu\phi. \tag{1.5}$$

The minimum of the potential spontaneously breaks the U(1) symmetry,

$$\langle\phi\rangle = \sqrt{\frac{\mu^2}{\lambda}} = \frac{v}{\sqrt{2}}\ . \tag{1.6}$$

We expand the scalar field around its expectation value

$$\phi = \frac{v + h + i\chi}{\sqrt{2}}\ . \tag{1.7}$$

The covariant derivative then is

$$D_\mu\phi = (\partial_\mu - ieA_\mu)\frac{v + h + i\chi}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\partial_\mu h + i\partial_\mu\chi - ie(v+h)A_\mu + e\chi A_\mu). \tag{1.8}$$

The kinetic term is therefore

$$\begin{aligned}
&(D_\mu\phi)^*(D^\mu\phi) \\
&= \frac{1}{2}\left|\partial_\mu h + i\partial_\mu\chi - ie(v+h)A_\mu + e\chi A_\mu\right|^2 \\
&= \frac{1}{2}\left[(\partial h)^2 + (\partial\chi)^2 + 2e\chi A_\mu\partial^\mu h - 2e\partial_\mu\chi(v+h)A^\mu + e^2((v+h)^2+\chi^2)A^2\right]
\end{aligned} \tag{1.9}$$

From the last term, we identify $m_A^2 = e^2v^2$. Note the mixing term $-ev\partial_\mu\chi A^\mu = -m_A\partial_\mu\chi A^\mu$ which needs to be removed to diagonalize the unperturbed Hamiltonian.

The potential term on the other hand is

$$\begin{aligned}
V &= -\frac{\lambda v^2}{2}\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \\
&= \frac{\lambda}{8}\left[-2v^2((v+h)^2+\chi^2) + ((v+h)^2+\chi^2)^2\right] \\
&= \frac{\lambda}{8}\left[(v+h)^2(-v^2+2vh+h^2) + 2((v+h)^2-v^2)\chi^2 + \chi^4\right] \\
&= \frac{\lambda}{8}\left[-v^4 + (2vh+h^2)^2 + 2(2vh+h^2)\chi^2 + \chi^4\right].
\end{aligned} \tag{1.10}$$

We idenfity the masses $m_h^2 = \lambda v^2$, $m_\chi^2 = 0$. The latter is expected for a Nambu–Godstone boson.

2 Gauge Fixing

To eliminate the mixing term between the gauge boson and the Nambu–Goldstone boson, we adopt the gauge-fixing term as

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi m_A\chi)^2 \tag{2.1}$$

The cross term $-\partial_\mu A^\mu m_A\chi$ cancels the mixing term from the kinetic term of the scalar field $-m_A\partial_\mu\chi A^\mu$ upon integration by parts. This form of the gauge-fixing term is called R_ξ-gauge, where R is meant to stand for "renormalizable."

In addition, we need the Faddev–Popov determinant. The infinitesimal gauge variation of the gauge-fixing term is

$$\delta(\partial_\mu A^\mu + \xi m_A \chi) = \partial_\mu \partial^\mu \theta + \xi m_A e(v+h)\theta. \tag{2.2}$$

Therefore, we need

$$\mathcal{L}_{FP} = -\bar{c}\left[\Box + \xi m_A^2 \left(1 + \frac{h}{v}\right)\right] c. \tag{2.3}$$

We identify $m_c^2 = m_{\bar{c}}^2 = \xi m_A^2$. To be consistent with the BRST symmetry discussed later, and keep the Lagrangian hermitian, we need to assign the hermiticity $c^\dagger = c$ (hermitian) and $\bar{c}^\dagger = -\bar{c}$ (anti-hermitian).*

By assembling the quadratic pieces of the Lagrangian, we find the unperturbed Lagrangian

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\chi)^2 + \partial_\mu\bar{c}\partial^\mu c \\
&\quad + \frac{1}{2}m_A^2 A_\mu A^\mu - \frac{1}{2}m_h^2 h^2 - \xi m_A^2 \bar{c}c - m_A \partial_\mu \chi A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu + \xi m_A \chi)^2 \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\chi)^2 + \partial_\mu\bar{c}\partial^\mu c \\
&\quad + \frac{1}{2}m_A^2 A_\mu A^\mu - \frac{1}{2}m_h^2 h^2 - \xi m_A^2 \bar{c}c - \frac{1}{2}\xi m_A^2 \chi^2.
\end{aligned} \tag{2.4}$$

It is easy to obtain the propagators for the FP ghost and the Nambu–Goldstone boson,

$$\frac{i}{p^2 - \xi m_A^2} \; . \tag{2.5}$$

The pole is at ξm_A^2, which is supposed to be the mass-squared for these fields. Since physical observables should not depend on the gauge-fixing parameter ξ, the Faddeev–Popov ghosts and Nambu–Goldston boson are *unphysical*.

To work out the propagator for the massive photon, we go to the momentum space for the kinetic term

$$\mathcal{L}_0 = -\frac{1}{4}(q_\mu A_\nu - q_\nu A_\mu)(q^\mu A^\nu - q^\nu A^\mu) - \frac{1}{2\xi}(q_\mu A^\mu)(q_\nu A^\nu) + \frac{1}{2}m_A^2 A_\mu A^\mu$$

*The best reference on confusing aspects of the BRST symmetry is the review article by Taichiro Kugo and Izumi Ojima, Progress of Theoretical Physics Vol. 60 No. 6 (1978) pp. 1869-1889

$$
\begin{aligned}
&= \frac{1}{2}A^\nu \left(-g_{\mu\nu}q^2 + q_\mu q_\nu - \frac{1}{\xi}q_\mu q_\nu + g_{\mu\nu}m_A^2\right) A^\mu \\
&= \frac{1}{2}A^\nu \left[\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)(q^2 - m_A^2) - \frac{1}{\xi}\frac{q_\mu q_\nu}{q^2}(q^2 - \xi m_A^2)\right] A^\mu . \qquad (2.6)
\end{aligned}
$$

By noting that two matrices

$$
\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right), \qquad \frac{q_\mu q_\nu}{q^2} \qquad (2.7)
$$

are both projection operators on orthogonal subspaces, we can invert the quadratic term of the Lagrangian and find the propagator

$$
\frac{i}{q^2 - m_A^2 + i\varepsilon}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i\xi}{q^2 - \xi m_A^2 + i\varepsilon}\frac{q_\mu q_\nu}{q^2} . \qquad (2.8)
$$

Note that the poles at $q^2 = m_A^2$ and ξm_A^2 come with the Feynman's prescription ($i\varepsilon$) but the projection operator does not have $i\varepsilon$ at $q^2 = 0$.

This form of the propagator means that for polarization states that satisfy $q_\mu \epsilon^\mu = 0$, the pole is at $q^2 = m_A^2$, while there is another pole at $q^2 = \xi m_A^2$ for the "scalar polarization" $\epsilon_s^\mu = q^\mu / m_A$.† Again, since physical observables should not depend on the gauge-fixing parameter ξ, the scalar polarization state is *unphysical*. Therefore, we find a *quartet* of unphysical states, ghost c, anti-ghost $\bar{c}$, Nambu–Goldstone boson χ, and scalar polarization of the vector boson. In any physical observables, contributions of this quartet must cancel among each other.

The so-called unitarity gauge is defined by the limit $\xi \to \infty$ where the propagator becomes the quadratic term of the Lagrangian and find the propagator

$$
\begin{aligned}
&\frac{i}{q^2 - m_A^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{-m_A^2}\frac{q_\mu q_\nu}{q^2} \\
&\quad = \frac{i}{q^2 - m_A^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} + \frac{q^2 - m_A^2}{m_A^2}\frac{q_\mu q_\nu}{q^2}\right) \\
&\quad = \frac{i}{q^2 - m_A^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_A^2}\right) \qquad (2.9)
\end{aligned}
$$

†The fact that the scalar polarization has the normalization $-\epsilon_s^2 = -\xi$ needs to be shown by carefully going through the canonical quantization in the Appendix. In the propagator, however, it is apparent due to the residue $-\xi$. Namely the state has negative norm $-\xi$.

This is nothing but the propagator for a massive vector boson (Proca field). However, this form has a very bad high-energy behavior compared to the form in the general R_ξ gauge for a finite ξ.

The so-called 't Hooft–Feynman gauge chooses $\xi = 1$ where the propagator is particularly simple,

$$\frac{i}{q^2 - m_A^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2 - m_A^2}\frac{q_\mu q_\nu}{q^2} = \frac{-i g_{\mu\nu}}{q^2 - m_A^2} \ . \tag{2.10}$$

On the other hand, the so-called Landau gauge chooses $\xi = 0$ and the propagator is also quite simple,

$$\frac{i}{q^2 - m_A^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \ . \tag{2.11}$$

In the literature, we often use the unitarity gauge if the calculations do not involve loops (tree-level) or the 't Hooft–Feynman gauge because the momentum-dependence is simple if they do involve loops.

3 BRST symmetry

The gauge-fixing term can be rewritten using an auxiliary field

$$\mathcal{L}_{gf} = B(\partial_\mu A^\mu + \xi m_A \chi) + \frac{\xi}{2}B^2. \tag{3.1}$$

Because this is quadratic in B, the path integral over B simply substitutes the stationary condition

$$\xi B = \partial_\mu A^\mu + \xi m_A \chi \tag{3.2}$$

and recovers the original form. Together with the Faddeev–Popov determinant, we find

$$\mathcal{L}_{gf+FP} = B(\partial_\mu A^\mu + \xi m_A \chi) + \frac{\xi}{2}B^2 - \bar{c}\left[\Box + \xi m_A^2\left(1 + \frac{h}{v}\right)\right] c. \tag{3.3}$$

We can now define the BRST symmetry. With an (infinitesimal) fermionic parameter η, it is given by the gauge transformation where the parameter is taken to be $\theta = \eta c$,

$$\delta_\eta \phi = ie\eta c\phi, \qquad \delta_\eta A_\mu = \eta \partial_\mu c. \tag{3.4}$$

Note that η is a global parameter and does not depend on spacetime positions. In addition, we need to regard it anti-hermitian $\eta^\dagger = -\eta$ to be consistent with the hermiticity of all the other fields.

The BRST transformation of the ghost and auxiliary fields are[‡]

$$\delta_\eta c = 0, \qquad \delta_\eta \bar{c} = \eta B, \qquad \delta_\eta B = 0. \tag{3.5}$$

It is easy to check that successive BRST transformations $\delta_\eta \delta_\chi = 0$ return zero.

Recalling Eq. (2.2), the BRST transformation of the gauge fixing condition is

$$\delta_\eta(\partial_\mu A^\mu + \xi m_A \chi) = (\partial_\mu \partial^\mu + \xi m_A e(v+h))\eta c. \tag{3.6}$$

Then the Lagrangian can be rewritten in a BRST-*exact* form,[§]

$$\mathcal{L}_{gf+FP} = \frac{1}{\eta}\delta_\eta\left[\bar{c}(\partial_\mu A^\mu + \xi m_A \chi) + \frac{\xi}{2}\bar{c}B\right]. \tag{3.7}$$

Because of the nilpotency of the BRST transformation $\delta^2 = 0$, we can see that this piece of the Lagrangian is BRST-exact, and hence BRST-invariant (closed).

4 Absence of Unphysical Degrees of Freedom

The quartet of (1) scalar polarization of the gauge boson, (2) the Nambu–Goldstone boson, (3) ghost and (4) anti-ghost all have mass-squared ξm_A^2 and hence are unphysical. They must, therefore, never show up in physical observables.

As an example of the cancellation among the quartet, let us compute their contribution to the decay width of the Higgs boson. We assume $4\xi m_A^2 < m_h^2$ to make it kinematically allowed. We identify the terms in the Lagrangian that contribute to this process,

$$\frac{1}{v}\left[m_A^2 h A_\mu A^\mu - \frac{m_h^2}{2}h\chi^2 - m_A(h\partial_\mu\chi - \chi\partial_\mu h)A^\mu - \xi m_A^2 \bar{c}c\right]. \tag{4.1}$$

[‡]Note we are dealing with an abelian gauge theory and hence $\{c, c\} = if^{abc}c^a c^b T^c = 0$.

[§]Here we borrow the terminology in cohomology. For instance in the de Rham cohomology of differential forms, the exterior derivative satisfies $d^2 = 0$. If a form ω_n satisfies $d\omega_n = 0$, it is said to be *closed*. If it can be written as $\omega_n = d\chi_{n-1}$, it is said to be *exact*. An exact form is always closed because of the nilpotency $d^2 = 0$.

We denote the four-momentum of the Higgs boson as p^μ ($p^2 = m_h^2$).

For actual computations, it is cumbersome to deal with the double fractions in the unphysical (scalar polarization) portion of the gauge boson propagator

$$\frac{-i\xi}{q^2 - \xi m_A^2 + i\varepsilon} \frac{q_\mu q_\nu}{q^2} \tag{4.2}$$

Note that the Feynman's prescription of $+i\varepsilon$ appears only for $q^2 = \xi m_A^2$ but not for the factor $1/q^2$, since it is only a part of the projection operator and hence should not contribute to the imaginary part. Therefore, it is easier to use the trick

$$\begin{aligned}
&\frac{i}{q^2 - m_A^2 + i\varepsilon}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i\xi}{q^2 - \xi m_A^2 + i\varepsilon}\frac{q_\mu q_\nu}{q^2} \\
&= \frac{-i g_{\mu\nu}}{q^2 - m_A^2 + i\varepsilon} + \frac{q_\mu q_\nu}{q^2}\left(\frac{i}{q^2 - m_A^2 + i\varepsilon} + \frac{-i\xi}{q^2 - \xi m_A^2 + i\varepsilon}\right) \\
&= \frac{-i g_{\mu\nu}}{q^2 - m_A^2 + i\varepsilon} + q_\mu q_\nu\left(\frac{i}{q^2(q^2 - m_A^2 + i\varepsilon)} + \frac{-i\xi}{q^2(q^2 - \xi m_A^2 + i\varepsilon)}\right) \\
&= \frac{-i g_{\mu\nu}}{q^2 - m_A^2 + i\varepsilon} \\
&\quad + q_\mu q_\nu\left(\frac{i}{m_A^2}\left[\frac{1}{q^2 - m_A^2 + i\varepsilon} - \frac{1}{q^2}\right] + \frac{-i\xi}{\xi m_A^2}\left[\frac{1}{q^2 - \xi m_A^2 + i\varepsilon} - \frac{1}{q^2}\right]\right) \\
&= \frac{-i g_{\mu\nu}}{q^2 - m_A^2 + i\varepsilon} + \frac{q_\mu q_\nu}{m_A^2}\left(\frac{i}{q^2 - m_A^2 + i\varepsilon} - \frac{i}{q^2 - \xi m_A^2 + i\varepsilon}\right).
\end{aligned} \tag{4.3}$$

With this rewriting, we do not have to worry about $1/q^2$ factors that would complicate the Feynman integrals.¶

4.1 Nambu–Goldstone Bosons

The two-point function for the Higgs boson due to the loop of the Nambu–Godstone boson is‖

$$i\Sigma_{\chi\chi}(p^2) = \frac{1}{2}\left(\frac{-i m_h^2}{v}\right)^2 \int \frac{d^D q}{(2\pi)^D} \frac{i}{q^2 - \xi m_A^2} \frac{i}{(q+p)^2 - \xi m_A^2} . \tag{4.4}$$

¶Thanks to Cory Schillaci who reminded me of this trick!

‖The sign convention here is that the full propagator is $i/(p^2 - m_h^2 + \Sigma)$ and hence $\Im m\Sigma = m_h \Gamma_h$.

The factor of a half is due to the fact that χ is identical to its anti-matter. Using the Feynman parameters, we rewrite it as (in $D = 4-2\epsilon$ dimension)**

$$\begin{aligned}
& i\Sigma_{\chi\chi}(p^2) \\
&= \frac{1}{2}\left(\frac{m_h^2}{v}\right)^2 \int \frac{d^D q}{(2\pi)^D} \int_0^1 dz \frac{1}{[q^2+2zq\cdot p+zp^2-\xi m_A^2]^2} \\
&= \frac{1}{2}\left(\frac{m_h^2}{v}\right)^2 \int \frac{d^D q}{(2\pi)^D} \int_0^1 dz \frac{1}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \\
&= \frac{1}{2}\left(\frac{m_h^2}{v}\right)^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(2)} [-z(1-z)p^2+\xi m_A^2 - i\varepsilon]^{-\epsilon} \\
&= \frac{1}{2}\left(\frac{m_h^2}{v}\right)^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(2)} \left(1-\epsilon\log[-z(1-z)p^2+\xi m_A^2 - i\varepsilon]\right)
\end{aligned} \tag{4.5}$$

The imaginary part of the two-point function is therefore

$$\Im m\Sigma_{\chi\chi}(m_h^2) = \frac{1}{2}\left(\frac{m_h^2}{v}\right)^2 \int_0^1 dz \frac{1}{(4\pi)^2} \pi\theta(z(1-z)m_h^2 - \xi m_A^2) \tag{4.6}$$

The positivity condition in the step function can be solved easily as

$$\frac{1-\beta}{2} < z < \frac{1+\beta}{2}, \quad \beta = \sqrt{1-\frac{4\xi m_A^2}{m_h^2}}\,. \tag{4.7}$$

Hence,

$$\Im m\Sigma_{\chi\chi}(m_h^2) = \frac{m_h^4}{v^2}\frac{\beta}{32\pi} = \frac{m_h^4}{v^2}\frac{\beta}{8\pi}\frac{1}{4}. \tag{4.8}$$

This result is consistent with the calculation of the decay width $h \to \chi\chi$,

$$\begin{aligned}
\Gamma(h\to\chi\chi) &= \frac{1}{2m_h}\frac{\beta}{8\pi}\frac{1}{2!}\int\frac{d\Omega}{4\pi}\left|\frac{m_h^2}{v}\right|^2 \\
&= \frac{\beta}{32\pi m_h}\frac{m_h^4}{v^2} \\
&= \frac{1}{m_h}\Im m\Sigma_{\chi\chi}(m_h^2).
\end{aligned} \tag{4.9}$$

Note the factor of 1/2! in the phase space integral because of two identical bosons in the final state.

**The Peskin–Schroeder uses $D = 4-\epsilon$, which ends up with cumbersome expressions with many $\epsilon/2$. My choice is more common in the literature.

4.2 Faddeev–Popov ghosts

The two-point function for the Higgs boson due to the loop of the Faddeev–Popov ghosts is

$$i\Sigma_{c\bar{c}}(p^2) = -\left(\frac{-i\xi m_A^2}{v}\right)^2 \int \frac{d^D q}{(2\pi)^D} \frac{i}{q^2 - \xi m_A^2} \frac{i}{(q+p)^2 - \xi m_A^2} \ . \tag{4.10}$$

Note the overall minus sign due to the Fermi statistics for the ghost fields. Following the same algebra, we find

$$\Im m \Sigma_{c\bar{c}}(m_h^2) = -\frac{\xi^2 m_A^4}{v^2} \frac{\beta}{16\pi} = -\frac{m_h^4}{v^2} \frac{\beta}{16\pi} \left(\frac{1-\beta^2}{4}\right)^2 = -\frac{m_h^4}{v^2} \frac{\beta}{8\pi} \frac{(1-\beta^2)^2}{32} \ . \tag{4.11}$$

This is consistent with the calculation of the decay width $h \to c\bar{c}$, except that we need to assign a negative norm to the state $|c\bar{c}\rangle$. This is so because the ghost and anti-ghost actually have an off-diagonal metric $\langle c|\bar{c}\rangle \neq 0$ if you go carefully through the canonical quantization. Adopting this rule, we find

$$\begin{aligned} \Gamma(h \to c\bar{c}) &= -\frac{1}{2m_h} \frac{\beta}{8\pi} \int \frac{d\Omega}{4\pi} \left| \frac{\xi m_A^2}{v} \right|^2 \\ &= -\frac{\beta}{16\pi m_h} \frac{\xi^2 m_A^4}{v^2} \\ &= \frac{1}{m_h} \Im m \Sigma_{c\bar{c}}(m_h^2). \end{aligned} \tag{4.12}$$

4.3 Mixed diagram with a Nambu–Goldston boson and a scalar polarization

The mixed χ-A_μ loop is

$$\begin{aligned} & i\Sigma_{\chi A}(p^2) \\ &= \left(\frac{m_A}{v}\right)^2 \int \frac{d^{4-2\epsilon} q}{(2\pi)^{4-2\epsilon}} (2p+q)_\mu \frac{i}{(p+q)^2 - \xi m_A^2} (-2p-q)_\nu \frac{q^\mu q^\nu}{m_A^2} \frac{-i}{q^2 - \xi m_A^2} \\ &= -\frac{1}{v^2} \int \frac{d^D q}{(2\pi)^D} \frac{(q^2 + 2p\cdot q)^2}{(p+q)^2 - \xi m_A^2} \frac{1}{q^2 - \xi m_A^2} \\ &= -\frac{1}{v^2} \int \frac{d^D q}{(2\pi)^D} \int_0^1 dz \frac{(q^2 + 2p\cdot q)^2}{[q^2 + 2(1-z)p\cdot q + (1-z)p^2 - \xi m_A^2]^2} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{v^2}\int \frac{d^Dq}{(2\pi)^D}\int_0^1 dz \frac{((q-(1-z)p)^2+2p\cdot(q-(1-z)p))^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \\
&= -\frac{1}{v^2}\int \frac{d^Dq}{(2\pi)^D}\int_0^1 dz \frac{(q^2+2zp\cdot q-(1-z^2)p^2)^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \\
&= -\frac{1}{v^2}\int \frac{d^Dq}{(2\pi)^D}\int_0^1 dz \frac{(q^2)^2-2(1-z^2)q^2p^2+4z^2(p\cdot q)^2+(1-z^2)^2(p^2)^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2}
\end{aligned}
\tag{4.13}
$$

We use the formula proven in the appendix,

$$
\int \frac{d^Dq}{(2\pi)^D}\frac{(q^2)^k}{[q^2-m^2-i\varepsilon]^n} = \frac{i(-1)^{n+k}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2}+k)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(\frac{D}{2})\Gamma(n)}[m^2]^{D/2+k-n}\ . \tag{4.14}
$$

We can perform d^Dq integral and find

$$
\begin{aligned}
&i\Sigma_{\chi A}(p^2) \\
&= -\frac{1}{v^2}\int \frac{d^Dq}{(2\pi)^D}\int_0^1 dz \frac{(q^2)^2-2(1-z^2)q^2p^2+4z^2(p\cdot q)^2+(1-z^2)^2(p^2)^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \\
&= -\frac{1}{v^2}\frac{i}{(4\pi)^{2-\epsilon}}\int_0^1 dz\left[\frac{\Gamma(4-\epsilon)\Gamma(-2+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)p^2+\xi m_A^2]^{2-\epsilon}\right. \\
&\quad +\frac{\Gamma(3-\epsilon)\Gamma(-1+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}\left(-2(1-z^2)+\frac{4z^2}{4-2\epsilon}\right)p^2[-z(1-z)p^2+\xi m_A^2]^{1-\epsilon} \\
&\quad \left.+\frac{\Gamma(\epsilon)}{\Gamma(2)}(1-z^2)^2(p^2)^2[-z(1-z)p^2+\xi m_A^2]^{-\epsilon}\right] \\
&= -\frac{1}{v^2}\frac{i}{(4\pi)^{2-\epsilon}}\int_0^1 dz\left[\frac{\Gamma(4-\epsilon)\Gamma(-2+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)p^2+\xi m_A^2]^{2}\right. \\
&\quad +\frac{\Gamma(3-\epsilon)\Gamma(-1+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}\left(-2(1-z^2)+\frac{4z^2}{4-2\epsilon}\right)p^2[-z(1-z)p^2+\xi m_A^2] \\
&\quad \left.+\frac{\Gamma(\epsilon)}{\Gamma(2)}(1-z^2)^2(p^2)^2\right][-z(1-z)p^2+\xi m_A^2-i\varepsilon]^{-\epsilon}.
\end{aligned}
\tag{4.15}
$$

The imaginary comes from the branch cut in the last factor,

$$
\begin{aligned}
&\Im m\Sigma_{\chi A}(m_h^2) \\
&= -\frac{1}{v^2}\frac{1}{(4\pi)^{2-\epsilon}}\int_0^1 dz\left[\frac{\Gamma(4-\epsilon)\Gamma(-2+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)m_H^2+\xi m_A^2]^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma(3-\epsilon)\Gamma(-1+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}\left(-2(1-z^2)+\frac{4z^2}{4-2\epsilon}\right)m_h^2[-z(1-z)m_h^2+\xi m_A^2] \\
&\left.+\frac{\Gamma(\epsilon)}{\Gamma(2)}(1-z^2)^2(m_h^2)^2\right](\epsilon\pi)\theta(z(1-z)m_h^2-\xi m_A^2) \\
= \quad & -\frac{1}{v^2}\frac{1}{(4\pi)^2}\int_0^1 dz\left[6\frac{1}{2\epsilon}[-z(1-z)m_H^2+\xi m_A^2]^2\right. \\
&+2\frac{-1}{\epsilon}\left(-2(1-z^2)+z^2\right)m_h^2[-z(1-z)m_h^2+\xi m_A^2] \\
&\left.+\frac{1}{\epsilon}(1-z^2)^2(m_h^2)^2\right](\epsilon\pi)\theta(z(1-z)m_h^2-\xi m_A^2) \\
= \quad & -\frac{1}{v^2}\frac{1}{16\pi}\int_0^1 dz\left[6\frac{1}{2}[-z(1-z)m_H^2+\xi m_A^2]^2\right. \\
&-2\left(-2(1-z^2)+z^2\right)m_h^2[-z(1-z)m_h^2+\xi m_A^2] \\
&\left.+(1-z^2)^2(m_h^2)^2\right]\theta(z(1-z)m_h^2-\xi m_A^2) \\
= \quad & -\frac{m_h^4}{v^2}\frac{1}{16\pi}\int_{(1-\beta)/2}^{(1+\beta)/2} dz\left[3\left[\left(z-\frac{1-\beta}{2}\right)\left(z-\frac{1+\beta}{2}\right)\right]^2\right. \\
&\left.+2\left(-2(1-z^2)+z^2\right)\left[\left(z-\frac{1-\beta}{2}\right)\left(z-\frac{1+\beta}{2}\right)\right]+(1-z^2)^2\right] \\
= \quad & -\frac{m_h^4}{v^2}\frac{1}{16\pi}\frac{9\beta+6\beta^5+\beta^5}{16}=-\frac{m_h^4}{v^2}\frac{\beta}{8\pi}\frac{(3+\beta^2)^2}{32}. \qquad (4.16)
\end{aligned}
$$

Here,

$$
\beta=\sqrt{1-\frac{4\xi m_A^2}{m_h^2}}\ .
$$

This is consistent with the calculation of the decay width $h\to c\bar{c}$, except that we need to assign a negative norm to the scalar polarization state $-\epsilon_s^2=-\xi$ which you find if you go carefully through the canonical quantization. Without loss of generality, we can always choose the reference frame such that

$$
\begin{aligned}
p^\mu &= (m_h,0,0,0), && (4.17)\\
q^\mu &= \frac{m_h}{2}(1,0,0,\beta), && (4.18)\\
\epsilon_s^\mu(q) &= \frac{m_h}{2m_A}(1,0,0,\beta). && (4.19)
\end{aligned}
$$

Then the decay amplitude is (note the definition of q^μ is the opposite from that in the loop diagram above)

$$\begin{aligned} i\mathcal{M} &= \frac{m_A}{v}(2p-q)\cdot\epsilon_s(q) = \frac{m_A}{v}\frac{m_h}{2}(3,0,0,-1\beta)\cdot\frac{m_h}{2m_A}(1,0,0,\beta) \\ &= \frac{m_h^2}{4v}(3+\beta^2). \end{aligned} \tag{4.20}$$

Adopting the rule with the negative norm, we find

$$\begin{aligned} \Gamma(h\to\chi A_s) &= -\frac{1}{2m_h}\frac{\beta}{8\pi}\int\frac{d\Omega}{4\pi}\frac{m_h^4}{16v^2}(3+\beta^2)^2 \\ &= -\frac{\beta}{16\pi m_h}\frac{m_h^4}{16v^2}(3+\beta^2)^2 \\ &= \frac{1}{m_h}\Im m\Sigma_{c\bar{c}}(m_h^2). \end{aligned} \tag{4.21}$$

4.4 Scalar Gauge Bosons

The two-point function for the Higgs boson due to the loop of the gauge boson of mass ξm_A^2 is

$$\begin{aligned} &i\Sigma_{AA}(p^2) \\ &= \frac{1}{2}\left(\frac{2im_A^2}{v}\right)^2\int\frac{d^Dq}{(2\pi)^D}\frac{-i}{q^2-\xi m_A^2}\frac{q_\mu q_\nu}{m_A^2}\frac{-i}{(p+q)^2-\xi m_A^2}\frac{(p+q)^\mu(p+q)^\nu}{m_A^2} \\ &= \frac{2}{v^2}\int\frac{d^Dq}{(2\pi)^D}\frac{(p\cdot q+q^2)^2}{q^2-\xi m_A^2}\frac{1}{(p+q)^2-\xi m_A^2} \\ &= \frac{2}{v^2}\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dz\frac{(p\cdot(q-zp)+(q-zp)^2)^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \\ &= \frac{2}{v^2}\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dz\frac{(q^2+(1-2z)p\cdot q-z(1-z)p^2)^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \end{aligned} \tag{4.22}$$

We use the formula proven in the appendix,

$$\int\frac{d^Dq}{(2\pi)^D}\frac{(q^2)^k}{[q^2-m^2-i\varepsilon]^n} = \frac{i(-1)^{n+k}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2}+k)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(\frac{D}{2})\Gamma(n)}[m^2]^{D/2+k-n}\,. \tag{4.23}$$

We can perform $d^D q$ integral and find

$$
\begin{aligned}
& i\Sigma_{AA}(p^2) \\
&= \frac{2}{v^2}\int \frac{d^D q}{(2\pi)^D}\int_0^1 dz \frac{(q^2)^2 - 2z(1-z)q^2p^2 + (1-2z)^2(p\cdot q)^2 + z^2(1-z)^2(p^2)^2}{[q^2+z(1-z)p^2-\xi m_A^2]^2} \\
&= \frac{2}{v^2}\frac{i}{(4\pi)^{2-\epsilon}}\int_0^1 dz \left[\frac{\Gamma(4-\epsilon)\Gamma(-2+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)p^2+\xi m_A^2]^{2-\epsilon}\right. \\
&\quad + \left(2z(1-z) - \frac{(1-2z)^2}{4-2\epsilon}\right)p^2\frac{\Gamma(3-\epsilon)\Gamma(-1+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)p^2+\xi m_A^2]^{1-\epsilon} \\
&\quad \left. + z^2(1-z)^2(p^2)^2\frac{\Gamma(\epsilon)}{\Gamma(2)}[-z(1-z)p^2+\xi m_A^2]^{-\epsilon}\right] \\
&= \frac{2}{v^2}\frac{i}{(4\pi)^{2-\epsilon}}\int_0^1 dz \left[\frac{\Gamma(4-\epsilon)\Gamma(-2+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)p^2+\xi m_A^2]^{2}\right. \\
&\quad + \left(2z(1-z) - \frac{(1-2z)^2}{4-2\epsilon}\right)p^2\frac{\Gamma(3-\epsilon)\Gamma(-1+\epsilon)}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)p^2+\xi m_A^2] \\
&\quad \left. + z^2(1-z)^2(p^2)^2\frac{\Gamma(\epsilon)}{\Gamma(2)}\right][-z(1-z)p^2+\xi m_A^2 - i\varepsilon]^{-\epsilon} \; . \qquad (4.24)
\end{aligned}
$$

The imaginary comes from the branch cut in the last factor,

$$
\begin{aligned}
& \Im m\Sigma_{AA}(m_h^2) \\
&= \frac{2}{v^2}\frac{1}{(4\pi)^2}\int_0^1 dz \left[\frac{\Gamma(4)\Gamma(-2+\epsilon)}{\Gamma(2)\Gamma(2)}[-z(1-z)m_h^2+\xi m_A^2]^2\right. \\
&\quad + \left(2z(1-z) - \frac{(1-2z)^2}{4}\right)m_h^2\frac{\Gamma(3)\Gamma(-1+\epsilon)}{\Gamma(2)\Gamma(2)}[-z(1-z)m_h^2+\xi m_A^2] \\
&\quad \left. + z^2(1-z)^2(m_h^2)^2\frac{\Gamma(\epsilon)}{\Gamma(2)}\right](\epsilon\pi)\theta(-z(1-z)m_h^2+\xi m_A^2) \\
&= \frac{2}{v^2}\frac{1}{(4\pi)^2}\int_0^1 dz \left[6\frac{1}{2\epsilon}[-z(1-z)m_h^2+\xi m_A^2]^2\right. \\
&\quad + \left(2z(1-z) - \frac{(1-2z)^2}{4}\right)m_h^2 2\frac{-1}{\epsilon}[-z(1-z)m_h^2+\xi m_A^2] \\
&\quad \left. + z^2(1-z)^2(m_h^2)^2\frac{1}{\epsilon}\right](\epsilon\pi)\theta(-z(1-z)m_h^2+\xi m_A^2) \\
&= \frac{2}{v^2}\frac{1}{16\pi}\int_{(1-\beta)/2}^{(1+\beta)/2} dz \left[3\left[\left(z-\frac{1-\beta}{2}\right)\left(z-\frac{1+\beta}{2}\right)\right]^2\right.
\end{aligned}
$$

$$
\begin{aligned}
& -2\left(2z(1-z)-\frac{(1-2z)^2}{4}\right)m_h^2\left(z-\frac{1-\beta}{2}\right)\left(z-\frac{1+\beta}{2}\right) \\
& +z^2(1-z)^2(m_h^2)^2\Big] \\
= \; & \frac{m_h^4}{8\pi v^2}\frac{\beta+2\beta^3+\beta^5}{16}=\frac{m_h^4}{v^2}\frac{\beta}{8\pi}\frac{(1+\beta^2)^2}{16}\ . \qquad (4.25)
\end{aligned}
$$

Here,

$$
\beta=\sqrt{1-\frac{4\xi m_A^2}{m_h^2}}\ .
$$

This is consistent with the calculation of the decay width $h \to A_s A_s$, except that we need to assign a negative norm to the scalar polarization state $-\epsilon_s^2 = -\xi$ which you find if you go carefully through the canonical quantization in the Appendix. Without loss of generality, we can always choose the reference frame such that

$$
\begin{aligned}
p^\mu &= (m_h, 0, 0, 0), && (4.26)\\
q^\mu &= \frac{m_h}{2}(1,0,0,\beta), && (4.27)\\
\epsilon_s^\mu(q) &= \frac{m_h}{2m_A}(1,0,0,\beta), && (4.28)\\
(p-q)^\mu &= \frac{m_h}{2}(1,0,0,-\beta), && (4.29)\\
\epsilon_s^\mu(p-q) &= \frac{m_h}{2m_A}(1,0,0,-\beta). && (4.30)
\end{aligned}
$$

Then the decay amplitude is (note the definition of q^μ is the opposite from that in the loop diagram above)

$$
i\mathcal{M}=\frac{2im_A^2}{v}\epsilon_s(q)\cdot\epsilon_s(p-q)=\frac{2im_A^2}{v}\frac{m_h^2}{4m_A^2}(1+\beta^2)=\frac{im_h^2}{2v}(1+\beta^2). \qquad (4.31)
$$

Adopting the rule with the negative norm for each scalar polarization, we find a positive contribution to the decay width

$$
\begin{aligned}
\Gamma(h\to A_sA_s) &= -\frac{1}{2m_h}\frac{\beta}{8\pi}\frac{1}{2!}\int\frac{d\Omega}{4\pi}\frac{m_h^4}{4v^2}(1+\beta^2)^2 \\
&= \frac{\beta}{16\pi m_h}\frac{m_h^4}{8v^2}(1+\beta^2)^2 \\
&= \frac{1}{m_h}\Im m\Sigma_{AA}(m_h^2). && (4.32)
\end{aligned}
$$

Note the factor of 1/2! in the phase space integral because of two identical bosons in the final state.

4.5 The sum

We simply sum up contributions in Eq. (4.8,4.11,4.16,4.25) and find

$$\begin{aligned}
&\Im m\Sigma_{\chi\chi+c\bar{c}+\chi A+AA} \\
&= \frac{m_h^4}{v^2}\frac{\beta}{8\pi}\left[\frac{1}{4}-\frac{(1-\beta^2)^2}{32}-\frac{(3+\beta^2)^2}{32}+\frac{(1+\beta^2)^2}{16}\right] \\
&= \frac{m_h^4}{v^2}\frac{\beta}{8\pi}\frac{8-(1-2\beta^2+\beta^4)-(9+6\beta^2+\beta^4)+(2+4\beta^2+2\beta^4)}{32} \\
&= 0. \qquad (4.33)
\end{aligned}$$

Indeed the contribution of unphysical states of mass ξm_A^2 all cancel out within the quartet of scalar polarization, Nambu–Goldstone boson, ghost, and anti-ghost, as required by the gauge invariance of the theory.

4.6 One physical, one unphysical

Having established the rules to compute the decay rates directly without relying on the one-loop diagrams, let us also verify the absence of $h \to A_\mu \chi$ and $h \to A_\mu A_s$ decays. It is easy to verify that the transverse polarizations do not contribute at all to either of them. Therefore we focus on the longitudinal gauge bosons $h \to A_L \chi$ and $A_L A_s$. We need to be aware that A_L has mass m_A while χ and A_s mass $\sqrt{\xi} m_A$. Without loss of generality, we can always choose the reference frame such that[††]

$$p^\mu = (m_h, 0, 0, 0), \qquad (4.34)$$

$$q^\mu = \frac{m_h}{2}(1+\frac{\xi m_A^2}{m_h^2}-\frac{m_A^2}{m_h^2}, 0, 0, \bar{\beta}), \qquad (4.35)$$

$$\epsilon_s^\mu(q) = \frac{m_h}{2m_A}(1+\frac{\xi m_A^2}{m_h^2}-\frac{m_A^2}{m_h^2}, 0, 0, \bar{\beta}), \qquad (4.36)$$

$$(p-q)^\mu = \frac{m_h}{2}(1-\frac{\xi m_A^2}{m_h^2}+\frac{m_A^2}{m_h^2}, 0, 0, -\bar{\beta}), \qquad (4.37)$$

[††]See general discussions of phase space in Note on Phase Space for 233B

$$
\begin{aligned}
\epsilon_L^\mu(p-q) &= \frac{m_h}{2m_A}(\bar\beta, 0, 0, -1 + \frac{\xi m_A^2}{m_h^2} - \frac{m_A^2}{m_h^2}), && (4.38)\\
\bar\beta &= \sqrt{1 - \frac{2(\xi m_A^2 + m_A^2)}{m_h^2} + \frac{(\xi m_A^2 - m_A^2)^2}{m_h^4}}\ . && (4.39)
\end{aligned}
$$

Then the decay amplitude $h \to A_L \chi$ is

$$
\begin{aligned}
i\mathcal{M} &= \frac{m_A}{v}(2p-q)\cdot\epsilon_L(p-q)\\
&= \frac{m_A}{v}\frac{m_h}{2}(3 - \frac{\xi m_A^2}{m_h^2} + \frac{m_A^2}{m_h^2}, 0, 0, -\bar\beta)\cdot\frac{m_h}{2m_A}(\bar\beta, 0, 0, -1 + \frac{\xi m_A^2}{m_h^2} - \frac{m_A^2}{m_h^2})\\
&= \frac{m_h^2}{4v}2\bar\beta. && (4.40)
\end{aligned}
$$

Therefore the partial width is

$$
\Gamma(h \to \chi A_L) = \frac{1}{2m_h}\frac{\bar\beta}{8\pi}\int\frac{d\Omega}{4\pi}\frac{m_h^4}{16v^2}4\bar\beta^2 = \frac{1}{2m_h}\frac{\bar\beta^3}{8\pi}\frac{m_h^4}{4v^2}\ . \qquad (4.41)
$$

On the other hand, the decay amplitude $h \to A_L A_s$ is

$$
\begin{aligned}
i\mathcal{M} &= \frac{2im_A^2}{v}\frac{m_h}{2m_A}(1 + \frac{\xi m_A^2}{m_h^2} - \frac{m_A^2}{m_h^2}, 0, 0, \bar\beta)\cdot\frac{m_h}{2m_A}(\bar\beta, 0, 0, -1 + \frac{\xi m_A^2}{m_h^2} - \frac{m_A^2}{m_h^2})\\
&= \frac{im_h^2}{2v}2\bar\beta. && (4.42)
\end{aligned}
$$

Adopting the rule with the negative norm for each scalar polarization, we find a positive contribution to the decay width

$$
\Gamma(h \to A_L A_s) = -\frac{1}{2m_h}\frac{\bar\beta}{8\pi}\int\frac{d\Omega}{4\pi}\frac{m_h^4}{16v^2}4\bar\beta^2 = -\frac{1}{2m_h}\frac{\bar\beta^3}{8\pi}\frac{m_h^4}{4v^2}\ . \qquad (4.43)
$$

It is plain that

$$
\Gamma(h \to \chi A_L) + \Gamma(h \to A_L A_s) = 0. \qquad (4.44)
$$

Therefore, unphysical states do not contribute to the physical observables such as the Higgs boson decay width.

4.7 Both physical

Having shown that all unphysical pieces are indeed not there, we'd like to compute the physical contribution to the Higgs width, too. Here we compute it both ways, one from one-loop diagrams, and the other from tree-level decay amplitudes.

The two-point function for the Higgs boson due to the loop of the gauge boson of mass m_A^2 is

$$
\begin{aligned}
& i\Sigma_{AA}(p^2) \\
&= \frac{1}{2}\left(\frac{2im_A^2}{v}\right)^2 \int \frac{d^Dq}{(2\pi)^D} \frac{-i}{q^2-m_A^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_A^2}\right) \\
&\qquad \frac{-i}{(p+q)^2-m_A^2}\left(g^{\mu\nu} - \frac{(p+q)_\mu (p+q)_\nu}{m_A^2}\right) \\
&= \frac{1}{2}\left(\frac{2m_A^2}{v}\right)^2 \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(p+q)^2-m_A^2} \frac{1}{q^2-m_A^2} \\
&\qquad \left(D - \frac{q^2}{m_A^2} - \frac{(p+q)^2}{m_A^2} + \frac{[q\cdot(p+q)]^2}{m_A^4}\right) \\
&= \frac{2}{v^2}\int \frac{d^Dq}{(2\pi)^D} \int_0^1 dz \frac{Dm_A^4 - (q^2+(p+q)^2)m_A^2 + [q\cdot(p+q)]^2}{[q^2+2zp\cdot q + zp^2 - m_A^2]^2} \\
&= \frac{2}{v^2}\int \frac{d^Dq}{(2\pi)^D} \int_0^1 dz \\
&\quad \frac{Dm_A^4 - (q-zp)^2 m_A^2 - (q+(1-z)p)^2 m_A^2 + [(q-zp)\cdot(q+(1-z)p)]^2}{[q^2+z(1-z)p^2-m_A^2]^2} \\
&= \frac{2}{v^2}\int \frac{d^Dq}{(2\pi)^D} \int_0^1 dz \frac{1}{[q^2+z(1-z)p^2-m_A^2]^2} \\
&\quad \left[Dm_A^4 - 2q^2 m_A^2 - [z^2+(1-z)^2]p^2 m_A^2 + (q^2)^2\right. \\
&\quad \left. +(1-2z)^2(q\cdot p)^2 - 2z(1-z)q^2p^2 + z^2(1-z)^2(p^2)^2\right]. \qquad (4.45)
\end{aligned}
$$

Because of the Lorentz invariance, $(q\cdot p)^2$ in the denominator is equivalent to q^2p^2/D. Therefore,

$$
\begin{aligned}
& i\Sigma_{AA}(m_h^2) \\
&= \frac{2}{v^2}\int \frac{d^Dq}{(2\pi)^D} \int_0^1 dz \frac{1}{[q^2+z(1-z)m_h^2-m_A^2]^2}
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg[Dm_A^4 - 2q^2 m_A^2 - [z^2+(1-z)^2]m_h^2 m_A^2 + (q^2)^2 \\
&+\frac{(1-2z)^2 - 2Dz(1-z)}{D} q^2 m_h^2 + z^2(1-z)^2 m_h^4\Bigg] \\
= \ & \frac{2}{v^2}\int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}}\frac{1}{\Gamma(2-\epsilon)\Gamma(2)}[-z(1-z)m_h^2 + m_A^2 - i\varepsilon]^{-\epsilon} \\
&\Bigg[\left(Dm_A^4 - [z^2+(1-z)^2]m_h^2 m_A^2 + z^2(1-z)^2 m_h^4\right)\Gamma(2-\epsilon)\Gamma(\epsilon) \\
&-\left(-2m_A^2 + \frac{(1-2z)^2 - 2Dz(1-z)}{D} m_h^2\right)\Gamma(3-\epsilon)\Gamma(-1+\epsilon) \\
&[-z(1-z)m_h^2 + m_A^2] + \Gamma(4-\epsilon)\Gamma(-2+\epsilon)[-z(1-z)m_h^2 + m_A^2]^2\Bigg]
\end{aligned}
\tag{4.46}
$$

The imaginary part comes from the branch cut in the $-\epsilon$ power,

$$
\begin{aligned}
&\Im m\Sigma_{AA}(m_h^2) \\
= \ & \frac{2}{v^2}\int_0^1 dz \frac{1}{(4\pi)^{2-\epsilon}}\frac{1}{\Gamma(2-\epsilon)\Gamma(2)}\epsilon\pi\theta(z(1-z)m_h^2 - m_A^2) \\
&\Bigg[\left(Dm_A^4 - [z^2+(1-z)^2]m_h^2 m_A^2 + z^2(1-z)^2 m_h^4\right)\Gamma(2-\epsilon)\Gamma(\epsilon) \\
&-\left(-2m_A^2 + \frac{(1-2z)^2 - 2Dz(1-z)}{D} m_h^2\right)\Gamma(3-\epsilon)\Gamma(-1+\epsilon) \\
&[-z(1-z)p^2 + m_A^2] + \Gamma(4-\epsilon)\Gamma(-2+\epsilon)[-z(1-z)p^2 + m_A^2]^2\Bigg] \\
= \ & \frac{2}{v^2}\int_{(1-\beta)/2}^{(1+\beta)/2} dz \frac{1}{(4\pi)^2}\pi \\
&\Bigg[\left(4m_A^4 - [z^2+(1-z)^2]m_h^2 m_A^2 + z^2(1-z)^2 m_h^4\right) \\
&+\left(-2m_A^2 + \frac{(1-2z)^2 - 8z(1-z)}{4} m_h^2\right) 2[-z(1-z)m_h^2 + m_A^2] \\
&+3[-z(1-z)m_h^2 + m_A^2]^2\Bigg] \\
= \ & \frac{m_h^4}{8\pi v^2}\left[\frac{3}{16}\beta^5 - \frac{1}{8}\beta^3 + \frac{3\beta}{16}\right] = \frac{m_h^4}{128\pi v^2}\beta(3 - 2\beta^2 + 3\beta^4).
\end{aligned}
\tag{4.47}
$$

This is consistent with the calculation of the decay width $h \to AA$. Without loss of generality, we can always choose the reference frame such that

$$\begin{aligned}
p^\mu &= (m_h, 0, 0, 0), && (4.48)\\
q^\mu &= \frac{m_h}{2}(1, 0, 0, \beta), && (4.49)\\
(p-q)^\mu &= \frac{m_h}{2}(1, 0, 0, -\beta), && (4.50)\\
\beta &= \sqrt{1 - \frac{4m_A^2}{m_h^2}}\ . && (4.51)
\end{aligned}$$

Then the decay amplitude is (note the definition of q^μ is the opposite from that in the loop diagram above)

$$i\mathcal{M} = \frac{2im_A^2}{v}\epsilon^*_{h_1}(q) \cdot \epsilon^*_{h_2}(p-q). \tag{4.52}$$

We use the relation

$$\sum_{h=-1}^{+1} \epsilon_h^\mu(q)\epsilon_h^{\nu *}(q) = -g^{\mu\nu} + \frac{q^\mu q^\nu}{m_A^2}\ . \tag{4.53}$$

Then we find

$$\begin{aligned}
\sum_{h_1 h_2} |\mathcal{M}|^2 &= \left(\frac{2m_A^2}{v}\right)^2 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_A^2}\right)\left(-g_{\mu\nu} + \frac{(p-q)_\mu (p-q)_\nu}{m_A^2}\right)\\
&= \left(\frac{2m_A^2}{v}\right)^2 \left(4 - 1 - 1 + \frac{[(q\cdot(p-q)]^2}{m_A^4}\right)\\
&= \left(\frac{2m_A^2}{v}\right)^2 \left(2 + \left[\frac{m_h^2}{4}(1+\beta^2)\right]^2 \frac{1}{m_A^4}\right)\\
&= \frac{4}{v^2}\left(2m_A^4 + \left[\frac{m_h^2}{4}(1+\beta^2)\right]^2\right)\\
&= \frac{4m_h^4}{v^2}\left(\frac{1}{8}(1-\beta^2)^2 + \left[\frac{1}{4}(1+\beta^2)\right]^2\right)\\
&= \frac{m_h^4}{4v^2}\left(2(1-\beta^2)^2 + (1+\beta^2)^2\right)\\
&= \frac{m_h^4}{4v^2}\left(3 - 2\beta^2 + 3\beta^4\right). && (4.54)
\end{aligned}$$

We find the contribution to the decay width

$$\begin{aligned}\Gamma(h \to AA) &= \frac{1}{2m_h}\frac{\beta}{8\pi}\frac{1}{2!}\int\frac{d\Omega}{4\pi}\frac{m_h^4}{4v^2}\left(3-2\beta^2+3\beta^4\right)\\ &= \frac{\beta}{16\pi m_h}\frac{m_h^4}{8v^2}\left(3-2\beta^2+3\beta^4\right)\\ &= \frac{1}{m_h}\Im m\Sigma_{AA}(m_h^2). \end{aligned} \tag{4.55}$$

Note the factor of 1/2! in the phase space integral because of two identical bosons in the final state.

Another way to compute the width is to compute the helicity amplitudes explicitly. For the kinematics above, the polarization vectors from Eqs. (B.10,B.11) are

$$\begin{aligned}\epsilon^\mu_\pm(q) &= \frac{1}{\sqrt{2}}(0,\mp 1,-i,0),\\ \epsilon^\mu_0(q) &= \frac{m_h}{2m_A}(\beta,0,0,1),\\ \epsilon^\mu_\pm(p-q) &= \frac{1}{\sqrt{2}}(0,\mp 1,+i,0),\\ \epsilon^\mu_0(p-q) &= \frac{m_h}{2m_A}(\beta,0,0,-1). \end{aligned} \tag{4.56}$$

Therefore the only non-vanishing amplitudes are

$$\begin{aligned}\mathcal{M}_{++} &= \frac{2m_A^2}{v}(-1),\\ \mathcal{M}_{--} &= \frac{2m_A^2}{v}(-1),\\ \mathcal{M}_{00} &= \frac{2m_A^2}{v}\frac{m_h^2}{4m_A^2}(\beta^2+1). \end{aligned} \tag{4.57}$$

Because this is a decay of a scalar particle, it makes sense that the helicities are the same for both the final-state photons to make sure there is no angular momentum. The helicity-summed squared amplitude is

$$\sum_{h_1,h_2}|\mathcal{M}|^2 = \left(\frac{2m_A^2}{v}\right)^2\left(1+1+\left[\frac{m_h^2}{4m_A^2}(\beta^2+1)\right]^2\right)$$

$$
\begin{aligned}
&= \frac{4m_A^4}{v^2}\left(2+\frac{m_h^4}{16m_A^4}(\beta^2+1)^2\right) \\
&= \frac{1}{4v^2}\left(32m_A^4+m_h^4(\beta^2+1)^2\right) \\
&= \frac{m_h^4}{4v^2}\left(2(1-\beta^2)^2+(\beta^2+1)^2\right) \\
&= \frac{m_h^4}{4v^2}\left(3-2\beta^2+3\beta^4\right), \qquad (4.58)
\end{aligned}
$$

verifying the previous result. This way, we can see explicitly that the longitudinal photons dominate when $m_h \gg m_A$.

Finally, we can test the equivalence theorem using this example. When $m_h \gg m_A$, the longitudinal polarization dominates and hence the decay width should become asymptotically the same as the width $\Gamma(h \to \chi\chi)$. Recalling the result from Eq. (4.9),

$$\Gamma(h \to \chi\chi) = \frac{\beta}{32\pi m_h}\frac{m_h^4}{v^2}, \qquad (4.59)$$

it is indeed the same as the asymptotic limit

$$\lim_{\beta\to 1}\Gamma(h \to AA) = \frac{1}{16\pi m_h}\frac{m_h^4}{8v^2}(3-2+3) = \frac{1}{16\pi m_h}\frac{m_h^4}{2v^2} \qquad (4.60)$$

verifying the equivalence theorem.

A Useful formula in dimensional regularization

It is useful to work out the general formula using the Wick rotation

$$
\begin{aligned}
&\int \frac{d^D q}{(2\pi)^D}\frac{(q^2)^k}{[q^2-m^2-i\varepsilon]^n} \\
&= i\int \frac{d^D q_E}{(2\pi)^D}\frac{(-1)^{n+k}(q_E^2)^k}{[q_E^2+m^2]^n} \\
&= i\int_0^\infty \frac{q_E^{D-2}dq_E^2}{(2\pi)^D}\frac{\pi^{D/2}}{\Gamma(\frac{D}{2})}\frac{(-1)^{n+k}(q_E^2)^k}{[q_E^2+m^2]^n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i(-1)^{n+k}\pi^{D/2}}{\Gamma(\frac{D}{2})(2\pi)^D}\int_0^\infty dq_E^2 \frac{(q_E^2)^{D/2+k-1}}{[q_E^2+m^2]^n} \\
&= \frac{i(-1)^{n+k}}{\Gamma(\frac{D}{2})(4\pi)^{D/2}}[m^2]^{D/2+k-n}\int_0^\infty dt \frac{t^{D/2+k-1}}{(t+1)^n} \\
&= \frac{i(-1)^{n+k}}{\Gamma(\frac{D}{2})(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2}+k)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(n)}[m^2]^{D/2+k-n} \\
&= \frac{i(-1)^{n+k}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2}+k)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(\frac{D}{2})\Gamma(n)}[m^2]^{D/2+k-n} \ . \qquad \text{(A.1)}
\end{aligned}
$$

It is easy to see that it reproduces the familiar result when $k=0$,

$$
\int \frac{d^Dq}{(2\pi)^D}\frac{1}{[q^2-m^2-i\varepsilon]^n} = \frac{i(-1)^n}{(4\pi)^{D/2}}\frac{\Gamma(-\frac{D}{2}+n)}{\Gamma(n)}[m^2]^{D/2-n} \ . \qquad \text{(A.2)}
$$

In addition, we can verify the recursion relation

$$
\begin{aligned}
&\int \frac{d^Dq}{(2\pi)^D}\frac{(q^2)^k - m^2(q^2)^{k-1}}{[q^2-m^2-i\varepsilon]^n} = \int \frac{d^Dq}{(2\pi)^D}\frac{(q^2)^{k-1}}{[q^2-m^2-i\varepsilon]^{n-1}} \\
&= \frac{i(-1)^{n+k}}{(4\pi)^{D/2}}[m^2]^{D/2+k-n} \\
&\quad \left[\frac{\Gamma(\frac{D}{2}+k)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(\frac{D}{2})\Gamma(n)} + \frac{\Gamma(\frac{D}{2}+k-1)\Gamma(-\frac{D}{2}-k+n+1)}{\Gamma(\frac{D}{2})\Gamma(n)}\right] \\
&= \frac{i(-1)^{n+k}}{(4\pi)^{D/2}}[m^2]^{D/2+k-n} \\
&\quad \left[\frac{((\frac{D}{2}+k-1)+(-\frac{D}{2}-k+n))\Gamma(\frac{D}{2}+k-1)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(\frac{D}{2})\Gamma(n-1)(n-1)}\right] \\
&= \frac{i(-1)^{n+k}}{(4\pi)^{D/2}}[m^2]^{D/2+k-n}\frac{\Gamma(\frac{D}{2}+k-1)\Gamma(-\frac{D}{2}-k+n)}{\Gamma(\frac{D}{2})\Gamma(n-1)} \ . \qquad \text{(A.3)}
\end{aligned}
$$

B Canonical Quantization

Because of the ghosts and negative-norm states, the canonical quantization of the gauge field is tricky, but nonetheless straightforward.

Yet, it is cumbersome to start from the canonical commutation relation to derive the mode expansion. Here we take a short cut in a heuristic way, working it out backwards. We focus on the quadratic part of the Lagrangian, and solve the equation of motion exactly to identify the modes. We then *guess* the commutation relations among mode operators (creation and annihilation operators) and verify the canonical commutation relation. This method may fail depending on the nature of interaction terms if they modify the canonical commutation relations. Fortunately we don't encounter this problem here.

B.1 Gauge Field

The quadratic part of the Lagrangian for the guage field together with the gauge-fixing term is

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}m_A^2 A_\mu A^\mu \\ &= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}m_A^2 A_\mu A^\mu. \end{aligned} \tag{B.1}$$

The equation of motion is

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu \partial_\mu A^\mu + m_A^2 A^\nu = 0. \tag{B.2}$$

Taking the divergence ∂_ν of the both sides, we find

$$\Box\frac{1}{\xi}\partial_\mu A^\mu + m_A^2 \partial_\nu A^\nu = 0. \tag{B.3}$$

Therefore, ∂A behaves as a free Klein–Gordon field of mass-squared ξm_A^2. We can write without a loss of generality

$$A^\mu(x) = A_\perp^\mu(x) + A_s^\mu(x), \tag{B.4}$$

where $\partial_\mu A_\perp^\mu(x) = 0$ thanks to the fact that the equation is linear in the field. The general solution for A_s is

$$A_s^\mu(x) = \int \frac{d^3p}{(2\pi)^3 2E_s}\epsilon_s^\mu(p_s)(a_s(p_s)e^{-ip_s\cdot x} + a_s^\dagger(p_s)e^{ip_s\cdot x}), \tag{B.5}$$

where

$$p_s^2 = \xi m_A^2, \qquad E_s = \sqrt{\vec{p}^2 + \xi m_A^2}\ , \qquad \epsilon_s^\mu(p_s) = \frac{p_s^\mu}{m_A}\ , \qquad \epsilon_s^2 = +\xi. \tag{B.6}$$

On the other hand, $A_\perp$ satisfies the equation of motion,

$$\partial_\mu F_\perp^{\mu\nu} + m_A^2 A_\perp^\nu = \Box A_\perp^\nu + m_A^2 A_\perp^\nu = 0. \tag{B.7}$$

Therefore $A_\perp$ satisfies the free Klein–Gordon equation of mass-squared m_A^2 subject to the constraint $\partial_\nu A_\perp^\nu = 0$. The general solution is then

$$A_\perp^\mu = \int \frac{d^3p}{(2\pi)^3 2E} \sum_{h=-1}^{+1} (\epsilon_h^\mu(p) a_h e^{-ip\cdot x} + \epsilon_h^{\mu *}(p) a_h^\dagger e^{ip\cdot x}). \tag{B.8}$$

It is convenient to choose the polarization vectors for helicity eigenstates for the four-momentum

$$p^\mu = E(1, \beta \sin\theta\cos\phi, \beta\sin\theta\sin\phi, \beta\cos\theta) \tag{B.9}$$

as

$$\epsilon_{h=\pm 1}^\mu = \frac{1}{\sqrt{2}}(\mp\epsilon_1^\mu - i\epsilon_2^\mu) \tag{B.10}$$

$$\epsilon_{h=0}^\mu = \frac{E}{m}(\beta, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{B.11}$$

where

$$\epsilon_1^\mu = (0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{B.12}$$

$$\epsilon_2^\mu = (0, -\sin\phi, \cos\phi, 0). \tag{B.13}$$

It is easy to verify

$$p\cdot\epsilon_h(p) = 0, \qquad \epsilon_h^*(p)\cdot\epsilon_{h'}(p) = -\delta_{h,h'}. \tag{B.14}$$

Now we *guess* the following commutation relations. We assume

$$[a_h(p), a_{h'}^\dagger(q)] = (2\pi)^3 2E\delta^3(\vec{p}-\vec{q})\delta_{h,h'}, \quad [a_s(p_s), a_s^\dagger(q_s)] = -(2\pi)^3 2E_s \delta^3(\vec{p}-\vec{q}). \tag{B.15}$$

while all other commutators vanish.

To obtain the canonical commutation relations, we first need to identify the canonical momenta conjugate to the field A^μ. For the spatial components, it is simply,

$$\pi^i = \frac{\partial\mathcal{L}_0}{\partial\dot{A}^i} = \dot{A}^i + \vec{\nabla}_i A^0 = -E^i. \tag{B.16}$$

Note that only $A_\perp$ contributes to E^i,

$$\pi^i = -E^i = -i\int \frac{d^3p}{(2\pi)^3 2E}\left[\sum_{h=\pm} E\epsilon^i_\pm(p)a_h e^{-ip\cdot x} + m_A\hat{p}^i a_{h=0}e^{-ip\cdot x} - c.c.\right], \tag{B.17}$$

where $\hat{p}^i = p^i/|\vec{p}|$.

On the other hand, the canonically conjugate momentum to A^0 comes only from the gauge-fixing term,

$$\pi^0 = \frac{\partial\mathcal{L}_0}{\partial\dot{A}^0} = -\frac{1}{\xi}(\partial_\mu A^\mu) = -\frac{1}{\xi}(\dot{A}^0 + \vec{\nabla}\cdot\vec{A}). \tag{B.18}$$

Note that only A_s contributes to π^0,

$$\pi^0 = im_A\int \frac{d^3p}{(2\pi)^3 2E_s}(a_s(p_s)e^{-ip_s\cdot x} - a_s^\dagger(p_s)e^{ip_s\cdot x}). \tag{B.19}$$

We also write out explicitly

$$A^0 = \int \frac{d^3p}{(2\pi)^3 2m_A}(a_s(p_s)e^{-ip_s\cdot x} + \beta a_{h=0}(p)e^{-ip\cdot x} + c.c.), \tag{B.20}$$

and

$$A^i(x) = \int \frac{d^3p}{(2\pi)^3 2m_A}\left[\beta_s\hat{p}^i a_s(p_s)e^{-ip_s\cdot x} + \frac{m_A}{E}\sum_{h=-1}^{+1}\epsilon^i_h(p)a_h e^{-ip\cdot x} + c.c.\right]. \tag{B.21}$$

Now we verify the canonical commutation relations.

$$\begin{aligned}
[A^0(\vec{x},0),\pi^0(\vec{y},0)] = & \int \frac{d^3p}{(2\pi)^3 2m_A}\int \frac{d^3q}{(2\pi)^3 2E_s(q)} \\
& [(a_s(p_s)e^{i\vec{p}_s\cdot\vec{x}} + a_s^\dagger(p_s)e^{-\vec{p}_s\cdot\vec{x}}), im_A(a_s(q_s)e^{i\vec{q}_s\cdot\vec{y}} - a_s^\dagger(q_s)e^{-i\vec{q}_s\cdot\vec{y}})] \\
= & \int \frac{d^3p}{(2\pi)^3 2m_A}\int \frac{d^3q}{(2\pi)^3 2E_s(q)} im_A 2E_s(2\pi)^3\delta^3(\vec{p}-\vec{q})(e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + c.c.) \\
= & \; i\delta^3(\vec{x}-\vec{y}).
\end{aligned} \tag{B.22}$$

Next,

$$[A^i(\vec{x},0),\pi^j(\vec{y},0)] = \int \frac{d^3p}{(2\pi)^3 2m_A}\int \frac{d^3q}{(2\pi)^3 2E}$$

$$
\begin{aligned}
&\Bigg[\left(\frac{m_A}{E}\sum_{\pm}\epsilon^i_{\pm}(p)a_{\pm}+\hat{p}^i a_h\right)e^{i\vec{p}\cdot\vec{x}}+c.c.,\\
&\quad i\left(\sum_{h=\pm}E\epsilon^{i*}_{\pm}(q)a_{\pm}+m_A\hat{q}^i a^{\dagger}_{h=0}\right)e^{-i\vec{q}\cdot\vec{y}}-c.c.\Bigg]\\
&= i\int\frac{d^3p}{(2\pi)^3 2m_A}\int\frac{d^3q}{(2\pi)^3 2E}m_A\left(\sum_{\pm}\epsilon^i_{\pm}(p)\epsilon^{i*}_{\pm}(q)+\hat{p}^i\hat{q}^i\right)\\
&\quad (e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot y)}+e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot y)})(2\pi)^3 2E\delta^3(\vec{p}-\vec{q})\\
&= i\delta^3(\vec{x}-\vec{y})\delta^{ij}. \qquad \text{(B.23)}
\end{aligned}
$$

We also need to verify the vanishing ones.

$$
\begin{aligned}
&[A^0(\vec{x},0),\pi^i(\vec{y},0)] = -i\int\frac{d^3p}{(2\pi)^3 2m_A}\int\frac{d^3 1}{(2\pi)^3 2E}\\
&\quad\left[\beta a_{h=0}(p)e^{i\vec{p}\cdot\vec{x}}+c.c., m_A\hat{q}^i a_{h=0}e^{i\vec{q}\cdot\vec{y}}-c.c.\right]\\
&= i\int\frac{d^3p}{(2\pi)^3 2m_A}\int\frac{d^3q}{(2\pi)^3 2E}2\beta m_A\hat{q}^i(e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}+c.c.)(2\pi)^3 2E\delta^3(\vec{p}-\vec{q})\\
&= i\int\frac{d^3p}{(2\pi)^3}\beta\hat{p}^i(e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+c.c.)=0. \qquad \text{(B.24)}
\end{aligned}
$$

The last equality is due to the inversion invariance of the measure $\vec{p}\to-\vec{p}$. Finally, we check

$$
\begin{aligned}
&[A^i(\vec{x},0),\pi^0(\vec{y},0)] = im_A\int\frac{d^3p}{(2\pi)^3 2m_A}\int\frac{d^3q}{(2\pi)^3 2E_s}\\
&\quad\left[\beta_s\hat{p}^i a_s(p_s)e^{i\vec{p}\cdot\vec{x}}+c.c., a_s(q_s)e^{i\vec{q}\cdot\vec{y}}-c.c.\right]\\
&= im_A\int\frac{d^3p}{(2\pi)^3 2m_A}\int\frac{d^3q}{(2\pi)^3 2E_s}\beta_s\hat{p}^i(e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}+c.c.)(2\pi)^3 2E_s\delta^3(\vec{p}-\vec{q})\\
&= i\int\frac{d^3p}{(2\pi)^3 2}\beta_s\hat{p}^i(e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+c.c.)=0. \qquad \text{(B.25)}
\end{aligned}
$$

Again in the last equality, the inversion invariance of the measure has been invoked.

Having verified the commutation relations, we study the Fock space. For the $h=\pm1,0$ gauge bosons, the commutation relations in Eq. (B.15) are the usual ones, and hence the usual Fock space. On the other hand, for the scalar gauge boson, we have the wrong sign commutation relation in Eq. (B.15). Therefore, the single-particle states are

$$|A_s, \vec{p}\rangle = a_s^\dagger(p_s)|0\rangle, \tag{B.26}$$
$$\langle A_s, \vec{p}|A_s, \vec{q}\rangle = -(2\pi)^3 2E_s \delta^3(\vec{p} - \vec{q}). \tag{B.27}$$

Namely that these are states with *negative* norms. A state with odd numbers of A_s would contribute *negatively* to the probability. Obviously such a contribution should be canceled by other unphysical degrees of freedom.

The main result of this discussion here is that for scalar polarization we assign the polarization vector

$$\epsilon_s^\mu = \frac{p_s^\mu}{m_A} \tag{B.28}$$

even though it is not normalized to unity $\epsilon_s^2 = +\xi$. In addition, we are supposed to make the probabilities negative if the state contains an odd number of scalar polarizations. This justifies the empirical treatment in Sections 4.4 and 4.6.

B.2 Faddeev–Popov ghosts

The terms of the Lagrangian quadratic in the ghost fields are

$$\mathcal{L}_0 = \partial_\mu \bar{c} \partial^\mu c - \xi m_A^2 \bar{c} c. \tag{B.29}$$

Obviously both the ghost and anti-ghost satisfy the Klein–Gordon equation,

$$(\Box + \xi m_A^2) c = (\Box + \xi m_A^2) \bar{c} = 0. \tag{B.30}$$

Recalling that c is hermitian and $\bar{c}$ anti-hermitian, the general solutions are

$$c(x) = \int \frac{d^3p}{(2\pi)^3 2E} (a(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}), \tag{B.31}$$
$$\bar{c}(x) = \int \frac{d^3p}{(2\pi)^3 2E} (\bar{a}(p) e^{-ip\cdot x} - \bar{a}^\dagger(p) e^{ip\cdot x}). \tag{B.32}$$

What is tricky is the canonical anti-commutation relations. First of all, when we identify the conjugate momenta for Grassmann-odd fields, let us take the convention that we differentiate the Lagrangian density by the time derivative of the field *from the right*. Namely,

$$\pi = \frac{\partial \mathcal{L}_0}{\partial \dot{c}} = \dot{\bar{c}}. \tag{B.33}$$

Using the same convention, the canonical momentum conjugate to $\bar{c}$ is

$$\pi = \frac{\partial \mathcal{L}_0}{\partial \dot{\bar{c}}} = -\dot{c} \tag{B.34}$$

because $\dot{\bar{c}}\dot{c} = -\dot{c}\dot{\bar{c}}$. The canonical anti-commutation relations are then

$$\{c(\vec{x},0), \dot{\bar{c}}(\vec{y},0)\} = \{\bar{c}(\vec{x},0), -\dot{c}(\vec{y},0)\} = i\delta^3(\vec{x}-\vec{y}), \tag{B.35}$$
$$\{c(\vec{x},0), \dot{c}(\vec{y},0)\} = \{\bar{c}(\vec{x},0), -\dot{\bar{c}}(\vec{y},0)\} = 0, \tag{B.36}$$
$$\{c(\vec{x},0), c(\vec{y},0)\} = \{\bar{c}(\vec{x},0), \bar{c}(\vec{y},0)\} = \{c(\vec{x},0), \bar{c}(\vec{y},0)\} = 0, \tag{B.37}$$
$$\{\dot{c}(\vec{x},0), \dot{c}(\vec{y},0)\} = \{\dot{\bar{c}}(\vec{x},0), \dot{\bar{c}}(\vec{y},0)\} = \{\dot{c}(\vec{x},0), \dot{\bar{c}}(\vec{y},0)\} = 0. \tag{B.38}$$

The first anti-commutation relation is

$$\begin{aligned} &\{c(\vec{x},0), \dot{\bar{c}}(\vec{y},0)\} \\ &= \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{d^3q}{(2\pi)^3 2E_q} \\ &\quad \left\{(a(p)e^{i\vec{p}\cdot\vec{x}} + a^\dagger(p)e^{-i\vec{p}\cdot\vec{x}}), -iE_q(\bar{a}(q)e^{i\vec{q}\cdot\vec{x}} + \bar{a}^\dagger(q)e^{-i\vec{q}\cdot\vec{x}})\right\} \\ &= i\delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{B.39}$$

The second one is

$$\begin{aligned} &\{\dot{c}(\vec{x},0), \bar{c}(\vec{y},0)\} \\ &= \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{d^3q}{(2\pi)^3 2E_q} \\ &\quad \left\{(-iE_p)(a(p)e^{i\vec{p}\cdot\vec{x}} - a^\dagger(p)e^{-i\vec{p}\cdot\vec{x}}), (\bar{a}(q)e^{i\vec{q}\cdot\vec{x}} - \bar{a}^\dagger(q)e^{-i\vec{q}\cdot\vec{x}})\right\} \\ &= -i\delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{B.40}$$

To satisfy them, we find

$$\{a(p), \bar{a}^\dagger(q)\} = \{a^\dagger(p), \bar{a}(q)\} = -(2\pi)^3 2E\delta^3(\vec{p}-\vec{q}). \tag{B.41}$$

Complementing them with

$$\begin{aligned}&\{a(p), a^\dagger(q)\} = \{\bar{a}^\dagger(p), \bar{a}(q)\} = \{a(p), a(q)\} = \{a^\dagger(p), a^\dagger(q)\}\\ &= \{\bar{a}(p), \bar{a}(q)\} = \{\bar{a}^\dagger(p), \bar{a}^\dagger(q)\} = \{a(p), \bar{a}(q)\} = \{a^\dagger(p), \bar{a}^\dagger(q)\} = 0,\end{aligned} \tag{B.42}$$

we find the rest of the canonical anti-commutation relations satisfied.

The above anti-commutation relations are *very bizarre*. Let us create a ghost

$$|c, \vec{p}\rangle = a^\dagger(p)|0\rangle. \tag{B.43}$$

It has a vanishing norm,

$$\langle c, \vec{p}|c, \vec{q}\rangle = \langle 0|a(p)a^\dagger(q)|0\rangle = \langle 0|\{a(p), a^\dagger(q)\}|0\rangle = 0. \tag{B.44}$$

The same is true with the anti-ghost state

$$|\bar{c}, \vec{p}\rangle = \bar{a}^\dagger(p)|0\rangle. \tag{B.45}$$

They, however, have a non-vanishing inner product

$$\langle \bar{c}, \vec{p}|c, \vec{q}\rangle = \langle 0|\bar{a}(p)a^\dagger(q)|0\rangle = \langle 0|\{\bar{a}(p), a^\dagger(q)\}|0\rangle = -(2\pi)^3 2E\delta^3(\vec{p} - \vec{q}). \tag{B.46}$$

We can diagonalize this off-diagonal metric and find that the states of the form $|c + \bar{c}\rangle$ have negative norm while the orthogonal combinations $|c - \bar{c}\rangle$ positive norm.

An interesting consequence of this normalization is what was studied in Section 4.2, the Higgs boson decaying into $c\bar{c}$ state. The final state is given by

$$|c, \vec{p}; \bar{c}, \vec{q}\rangle = a^\dagger(p)\bar{a}^\dagger(q)|0\rangle. \tag{B.47}$$

Its norm is

$$\begin{aligned}\langle c, \vec{p}'; \bar{c}, \vec{q}'|c, \vec{p}; \bar{c}, \vec{q}\rangle &= \langle 0|\bar{a}(q')a(p')a^\dagger(p)\bar{a}^\dagger(q)|0\rangle\\ &= -\langle 0|\bar{a}(q')a^\dagger(p)a(p')\bar{a}^\dagger(q)|0\rangle\\ &= -\langle 0|\{\bar{a}(q'), a^\dagger(p)\}\{a(p'), \bar{a}^\dagger(q)\}|0\rangle\\ &= -(2\pi)^3 2E\delta^3(\vec{p} - \vec{q}')(2\pi)^3 2E\delta^3(\vec{p}' - \vec{q}).\end{aligned} \tag{B.48}$$

Therefore, this state has a negative norm and hence we assign negative probabilty, as it was done in Section 4.2.