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Special Relativity and Electromagnetism

In the following reconstruction of the theory of electromagnetism in special relativity, we regard spacetime as the vector space $\mathbf{R}^4$ with the usual metric:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.1}$$

1 Electromagnetic Field

The electromagnetic field at any point in $\mathbf{R}^4$ is described by the $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & -B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{1.1}$$

We observe that F is antisymmetric, i.e. $F_{\alpha\beta} = -F_{\beta\alpha}$, which implies $F_{\alpha\alpha} = 0$. Define the structure constant $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ tensor:

$$S_{\alpha\beta\gamma} = \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} \tag{1.2}$$

The cyclic permutations of $S_{\alpha\beta\gamma}$ have the same value, and $S_{\alpha\gamma\beta} = -S_{\alpha\beta\gamma}$, since F is antisymmetric. Hence, switching any pair of indices produces a tensor that is the negative of the original tensor. This means S is also antisymmetric. Next, we consider the case where α, β, and γ are not distinct. Without loss of generality, assuming $\alpha = \beta$, and from the antisymmetry of S, we have:

$$S_{\alpha\alpha\gamma} = -S_{\alpha\alpha\gamma} = 0 \tag{1.3}$$

Thus, all components of S are zero except for those which have distinct indices.

2 Maxwell Equations

Let the electromagnetic potential be the one-form

$$A = A_\alpha \theta^\alpha, \qquad A_\alpha = (\phi, A_1, A_2, A_3) \tag{2.1}$$

and the electromagnetic current, which is the vector

$$J = J^\alpha e_\alpha, \qquad J^\alpha = (\rho, J^1, J^2, J^3) \tag{2.2}$$

Let the right hand side of (1.2) be zero, and look at the components of the equation

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha \tag{2.3}$$

When $\alpha = 0$, and $\{\beta, \gamma\} \subset \{1, 2, 3\}$, recalling the antisymmetry of S, the equations are as followed:

$$\begin{aligned}
\partial_j F_{0i} + \partial_0 F_{ij} + \partial_i F_{j0} &= \partial_0 B_i + \epsilon_{ijk}\partial_j E_k = 0 && (2.4)\\
&= \partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 && (2.5)\\
\partial_i F^{0i} &= 4\pi J^0 && (2.6)\\
&= \vec{\nabla} \cdot \vec{E} = 4\pi\rho && (2.7)
\end{aligned}$$

Likewise, when $\{\alpha, \beta, \gamma\} \subseteq \{1, 2, 3\}$, then:

$$\partial_i B_i = \vec{\nabla} \cdot \vec{B} = 0 \tag{2.8}$$

If $\alpha \in \{1, 2, 3\}$ and $\beta = 0$, then:

$$\begin{aligned}
\partial_0 F^{i0} + \partial_j F^{ij} &= 4\pi J^i && (2.9)\\
&= -\partial_t \vec{E} + \vec{\nabla} \times \vec{B} = 4\pi \vec{J} && (2.10)
\end{aligned}$$

One now has the set of Maxwell's equations:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 4\pi\rho && (2.11)\\
\vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0 && (2.12)\\
\vec{\nabla} \cdot \vec{B} &= 0 && (2.13)\\
\vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= 4\pi \vec{J} && (2.14)
\end{aligned}$$

To find the expressions for the fields $\vec{E}$ and $\vec{B}$ in terms of the potentials ϕ and $\vec{A}$, consider the components of the differential equation

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{2.15}$$

Again, we look at the separate cases where only one index of F is zero and where neither indices are zero, and we obtain:

$$F_{i0} = E_i = \partial_i A_0 - \partial_0 A_i \tag{2.16}$$

which is the equivalence of:

$$\vec{E} = \vec{\nabla} \cdot \phi - \partial_t \vec{A} \tag{2.17}$$

and

$$\begin{aligned}
F_{ij} = B_k &= \partial_i A_j - \partial_j A_i && (2.18)\\
&= \epsilon_{ijk} \partial_i A_j && (2.19)
\end{aligned}$$

which implies

$$\vec{B} = \vec{\nabla} \times \vec{B} \tag{2.20}$$

Furthermore, in 4-dimensional spacetime, if equation (2.15) is assumed and substituted into (1.2), then after distributing the derivatives (which are commutative), all terms cancel with one another:

$$\begin{aligned} S_{\alpha\beta\gamma} &= \partial_\gamma(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + \partial_\alpha(\partial_\beta A_\gamma - \partial_\gamma A_\beta) + \partial_\beta(\partial_\gamma A_\alpha - \partial_\alpha A_\gamma) & (2.21) \\ &= \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0 & (2.22) \end{aligned}$$

This identity is sometimes referred to as the Bianchi identity of electromagnetism.

It can also be shown that by substituting equation (2.15) into (2.3), with the appropriate index raising by the usual metric, one obtains the Maxwell equation for the potentials A_α.

When $\alpha = 0$, the result is another form of Gauss's Law (Eq. (2.11))

$$\begin{aligned} \partial_\beta F^{0\beta} &= \partial_\beta(\partial^0 A^\beta - \partial^\beta A^0) & (2.23) \\ &= -\partial_t \vec{\nabla} \cdot \phi + \nabla^2 \phi = 4\pi\rho & (2.24) \end{aligned}$$

And when α is a spatial coordinate, β ranges over both the spatial and temporal coordinates, we get the Ampère-Maxwell law (Eq. (2.14))

$$\partial_\beta(\partial^i A^\beta - \partial^\beta A^i) = -\vec{\nabla} \cdot \partial_t \phi + \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + \partial_t^2 \vec{A} = 4\pi \vec{J} \tag{2.25}$$

3 Gauge Transformation

Recalling that the electric and magnetic fields are differential forms of the scalar and vector potentials, ϕ and $\vec{A}$ are not uniquely determined. For each pair of $\vec{E}$ and $\vec{B}$ there exist at least two possible sets of A_α that can describe the same fields. With any given A_α, let χ be an arbitrary scalar function, we can perform the so-called gauge transformation on the potential

$$A'_\alpha = A_\alpha + \partial_\alpha \chi \tag{3.1}$$

Consequently, the new field is obtained from the potential in the same manner described by (2.15)

$$F'_{\alpha\beta} = \partial_\alpha A'_\beta - \partial_\beta A'_\alpha \tag{3.2}$$

It can be shown that the electromagnetic fields are gauge-invariant, i.e. the fields are unaffected by the potential transformation. By substituting (3.1) into (3.2) and distributing the derivatives, the terms with χ cancel out one another, leaving

$$F'_{\alpha\beta} = F_{\alpha\beta} \tag{3.3}$$

This result demonstrate the freedom in imposing extra conditions on ϕ and $\vec{A}$ and keeping $\vec{E}$ and $\vec{B}$ fixed. For example, if the gauge condition is

$$\partial^\alpha A'_\alpha = 0 \tag{3.4}$$

Then upon assuming Duhamel's principle of nonhomogeneous wave equation (it is always possible to solve a wave equation with an arbitrary but specified source), one can find a solution of the scalar form χ for the following equation

$$\begin{aligned} \partial^\alpha A'_\alpha &= \partial^\alpha A_\alpha + \partial^\alpha \partial_\alpha \chi & (3.5) \\ &= \quad \chi + \partial^\alpha A_\alpha = 0 & (3.6) \end{aligned}$$

where $\quad = \partial^\alpha \partial_\alpha = \nabla^2 - \partial_t^2$. Thus, there always exists a function to satisfy the given gauge condition, and in general any gauge condition.

Let us write out the Maxwell equation for A'_α

$$\begin{aligned} \partial_\beta(\partial^\alpha A'^\beta - \partial^\beta A'^\alpha) &= 4\pi J^\alpha & (3.7) \\ &= \partial_\beta(\partial^\alpha(A^\beta + \partial^\beta \chi) - \partial^\beta(A^\alpha + \partial^\alpha \chi)) & (3.8) \\ &= \partial^\alpha(\quad \chi + \partial_\beta A^\beta) - \quad (A^\alpha + \partial^\alpha \chi) & (3.9) \end{aligned}$$

The first term in (3.9) vanishes by (3.6). Therefore:

$$\quad A'_\alpha = -4\pi J_\alpha \tag{3.10}$$

The equation is again another nonhomogeneous wave equation that can always be solved for any arbitrary but specified current J^α.

4 Conservation of Charge

Let us look at the components of the equation

$$\partial_\alpha J^\alpha = 0 \tag{4.1}$$

With the usual method of separating the spatial components from the temporal one, one gets

$$\vec{\nabla} \cdot \vec{J} + \partial_t \rho = 0 \tag{4.2}$$

which implies that the rate of change of the charge density is indeed the magnitude of the current going through a system. This is called the principle of charge conservation.

One can substitute the expression of $F^{\alpha\beta}$ in terms of A^α into (2.3) and take the derivative of both sides to get (4.1), since α and β are dummy indices which can be relabeled.

$$\partial_\alpha \partial_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = 4\pi \partial_\alpha J^\alpha = 0 \tag{4.3}$$

Therefore, if (2.3) is satisfied, then the charge conservation law is automatically satisfied.

5 The Maxwell Energy-Momentum-Stress Tensor

In this section we will look at conservation laws of electromagnetic energy and momentum. In vacuum, the Lagrangian density for the electromagnetic field is the function

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^{\gamma\delta}F_{\gamma\delta} &(5.1)\\
&= -\frac{1}{4}\eta^{\alpha\gamma}\eta^{\beta\delta}F^{\alpha\beta}F_{\gamma\delta} &(5.2)\\
&= -\frac{1}{4}(F_{ij}F_{ij} + 2F_{\mu 0}F_{0\mu}) &(5.3)\\
&= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) &(5.4)\\
&= \frac{1}{2}[(\vec{\nabla}\cdot\phi - \partial_t\vec{A})^2 - (\vec{\nabla}\times\vec{A})^2] &(5.5)
\end{aligned}$$

Equation (5.4) shows that the Lagrangian density can be interpreted as the the difference between the kinetic energy $\frac{1}{2}|\vec{E}|^2$ and the potential energy $\frac{1}{2}|\vec{B}|^2$.

As an analogy to the Hamiltonian in classical mechanics, we can construct the Maxwell energy-momentum-stress tensor $T^{\alpha\beta}$ from the Lagrangian density above.

$$T^{\alpha\beta} = \frac{1}{4\pi}(F^{\alpha\gamma}F^{\beta}_{\gamma} - \frac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}) \qquad (5.6)$$

We note that α and β are interchangable, which makes T a symmetric tensor. Writing out the spatial and temporal components separately

$$\begin{aligned}
T^{00} &= \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2) \quad \text{“energy density”} &(5.7)\\
T^{i0} &= \frac{1}{4\pi}(\vec{E}\times\vec{B})_i \quad \text{“momentum density”} &(5.8)\\
T^{0i} &= \frac{1}{4\pi}(\vec{E}\times\vec{B})_i \quad \text{“energy current”} &(5.9)\\
T^{ij} &= \frac{1}{4\pi}[-(E_iE_j + B_iB_j) + \delta_{ij}\mathcal{L}] \quad \text{“momentum current”} &(5.10)
\end{aligned}$$

Next let us take the divergence of T, using the equation

$$\nabla_\sigma T^{\alpha\beta} = T^{\alpha\beta}_{,\sigma} + T^{\delta\beta}\Gamma^{\alpha}_{\delta\sigma} + T^{\alpha\delta}\Gamma^{\beta}_{\delta\sigma} \qquad (5.11)$$

where $\Gamma^{\alpha}_{\delta\sigma}$ is the Christoffel symbol.

Since the metric is independent of the electromagnetic field,

$$\begin{aligned}
\nabla_\sigma T^{\alpha\beta} &= \partial_\sigma T^{\alpha\beta} &(5.12)\\
&= \frac{1}{4\pi}\partial_\sigma(F^{\alpha\gamma}F^{\beta}_{\gamma} - \frac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}) &(5.13)
\end{aligned}$$

One can use the Bianchi identity (Eqn. (2.22)) and the Maxwell equations (Eqn. (2.3)) to simplify the above equation

$$\partial_\sigma T^{\alpha\beta} = -J_\beta F^{\alpha\beta} \tag{5.14}$$

This reduction demonstrates that the divergence of the energy-momentum-stress tensor is dependent on electromagnetic current and is zero if and only if the current is zero. That means the electromagnetic energy-momentum is not conserved if the current is nonzero. The reason for that is we have ignored the energy-momentum of the charged particle producing the current.

6 Energy-Momentum of Charged Particles

First, we will study the motion of one charged particle which has charge q, mass m, and is moving on the parametrized path $x^\alpha(\tau)$ where τ is the proper time. The particle has a unit timelike 4-velocity

$$U^\alpha = \frac{\partial x^\alpha}{\partial \tau} = (\gamma, \gamma v^1, \gamma v^2, \gamma v^3) \tag{6.1}$$

where $\gamma = \frac{1}{\sqrt{1-|\vec{v}|^2}}$, and a 4-momentum

$$p^\alpha = mU^\alpha \tag{6.2}$$

Let us look at the components of the equation

$$\begin{aligned} U^\beta \partial_\beta p^\alpha &= qU^\beta F^\alpha{}_\beta && (6.3)\\ \frac{\partial x^\beta}{\partial \tau}\partial_\beta p^\alpha &= qU^\beta \eta^{\delta\alpha} F_{\delta\beta} && (6.4)\\ &= \frac{\partial p^\alpha}{\partial \tau} && (6.5) \end{aligned}$$

from which we get the Lorentz force law

$$\partial_\tau p^i = q\gamma(E^i + (\vec{v}\times\vec{B})^i) \tag{6.6}$$

and the Lorentz power law

$$\partial_\tau p^0 = q\gamma(\vec{v}\cdot\vec{E}) \tag{6.7}$$

In general, consider a fluid of charged particles of rest mass m and charge q, with fluid velocity U^α and energy density ρ in the instantaneous local rest frame. Then the charge density in the instantaneous local rest frame is $\frac{q}{m}\rho$ and the electromagnetic current is

$$J^\alpha = \frac{q}{m}\rho U^\alpha \tag{6.8}$$

Assuming the only interaction between the particles in the fluid is electromagnetism, then

(i) the energy-momentum tensor for the fluid is the same as the one for dust:

$$T^{\alpha\beta}_{fluid} = \rho U^\alpha U^\beta, \tag{6.9}$$

(ii) each particle moves according to the Lorentz force equation:

$$U^\beta \partial_\beta (mU^\alpha) = qU^\beta F^\alpha_\beta, \tag{6.10}$$

(iii) the energy-momentum tensor for the electromagnetic field is:

$$T^{\alpha\beta}_{em} = \frac{1}{4\pi}(F^{\alpha\gamma}F^\beta_\gamma - \frac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F^{\gamma\delta}), \tag{6.11}$$

(iv) and the electromagnetic field satisfies the Bianchi identities and the Maxwell equations with current J^α.

Then, as seen in section 4, the electromagnetic current is conserved:

$$\partial_\alpha J^\alpha = 0 \tag{6.12}$$

and, similar to (5.14), the electromagnetic energy-momentum tensor satisfies

$$\partial_\beta T^{\alpha\beta}_{em} = -J_\alpha F^{\alpha\beta}. \tag{6.13}$$

If we factor $T^{\alpha\beta}_{fluid}$ as $(U^\beta)(\rho U^\alpha)$ (since T is symmetric) and apply product rule of differentiation to the energy-momentum tensor for the fluid, then we get:

$$\partial_\beta T^{\alpha\beta}_{fluid} = (\partial_\beta U^\beta)\rho U^\alpha + U^\beta \partial_\beta(\rho U^\alpha) \tag{6.14}$$

Recalling the relation between the electromagnetic current and the fluid velocity, the first term on the right hand side of the above equation vanishes due to conservation of the electromagnetic current. From the Lorentz force equation (6.10):

$$\begin{aligned} \partial_\beta T^{\alpha\beta}_{fluid} &= \frac{q\rho}{m}U^\beta F^\alpha_\beta && (6.15) \\ &= J_\beta F^{\alpha\beta} && (6.16) \\ & && (6.17) \end{aligned}$$

Thus the total energy-momentum is conserved:

$$\partial_\beta (T^{\alpha\beta}_{fluid} + T^{\alpha\beta}_{em}) = 0. \tag{6.18}$$

7 Rotations and Lorentz Boosts

In this section we will study the electromagnetic field under rotations and Lorentz boosts. Under a general Lorentz transformation, $\Lambda^{\alpha'}_\gamma$, the electromagnetic field transforms according to

$$F_{\alpha'\beta'} = F_{\gamma\delta}(\Lambda^{-1})^\gamma_{\alpha'}(\Lambda^{-1})^\delta_{\beta'}. \tag{7.1}$$

We then write

$$F_{\gamma\delta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & -B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{7.2}$$

and

$$F_{\alpha'\beta'} = \begin{pmatrix} 0 & -E_{1'} & -E_{2'} & -E_{3'} \\ E_{1'} & 0 & B_{3'} & -B_{2'} \\ E_{2'} & -B_{3'} & 0 & -B_{1'} \\ E_{3'} & B_{2'} & -B_{1'} & 0 \end{pmatrix} \tag{7.3}$$

First we assume that the Lorentz transformation is a rotation about the z-axis:

$$\Lambda^{\alpha'}_{\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \tag{7.4}$$

where

$$R^{i'}_{j} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.5}$$

Notice that $\Lambda^{-1} = \Lambda^T$. To see how the electric field transforms, let us look at $F_{i'0'}$

$$\begin{aligned} F_{i'0'} &= F_{\gamma\delta}(\Lambda^T)^{\gamma}_{i'}(\Lambda^T)^{\delta}_{0'} && (7.6) \\ &= F_{\gamma 0}(\Lambda^T)^{\gamma}_{i'} && (7.7) \\ &= F_{j0}(\Lambda^T)^{j}_{i'} && (7.8) \\ &= F_{j0}R^{i'}_{j} && (7.9) \\ & && (7.10) \end{aligned}$$

We could substitute $\gamma = i$ and $\delta = 0$ in the above equations because all other terms of Λ vanish. Similarly, the transformed magnetic field is found by transforming each of its component using (7.1). The choice of rotation matrix R was arbitrary, and the transformation was clearly independent of the axis of rotation. Therefore, we can claim that in general, the electric and magnetic fields transform like vectors under Lorentz transformation:

$$\begin{aligned} E^{i'} &= R^{i'}_{j}E^{j} && (7.11) \\ B^{i'} &= R^{i'}_{j}B^{j} && (7.12) \end{aligned}$$

Next we assume that the Lorentz transformation is a boost in the z-direction with velocity $\vec{v} = v\hat{e}_z$:

$$\Lambda^{\alpha'}_{\gamma} = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix} \tag{7.13}$$

where $\cosh\lambda = \gamma = \frac{1}{\sqrt{1-v^2}}$ and $\sinh\lambda = \gamma v = \frac{v}{\sqrt{1-v^2}}$. Again, using (7.1) and after a little bit of algebra, we can write the transformed electric and magnetic fields in terms of the original fields and λ as follows:

$$\vec{E}' = \begin{pmatrix} E^1\cosh\lambda + B^2\sinh\lambda \\ E^2\cosh\lambda - B^1\sinh\lambda \\ E^3 \end{pmatrix} \tag{7.14}$$

$$\vec{B}' = \begin{pmatrix} B^1\cosh\lambda - E^2\sinh\lambda \\ B^2\cosh\lambda + E^1\sinh\lambda \\ B^3 \end{pmatrix} \tag{7.15}$$

8 Lagrangian and Hamiltonian Formulations

Recall that in classical particle mechanics, the Lagrangian is

$$L = T - V = \frac{1}{2}m|\vec{v}|^2 - V(\vec{v}) \tag{8.1}$$

In discussing this Lagrangian, it is useful to regard $\vec{x}$ and $\vec{v} = \frac{d\vec{x}}{dt}$ as independent variables. One then computes

$$\frac{\partial L}{\partial x^i} = \partial_i L = \partial_i V \tag{8.2}$$

$$p_i = \frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \tag{8.3}$$

(The quantity p_i is called the momentum conjugate to x^i.) Then the Euler-Lagrange equation for this Lagrangian is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \partial_i L = \dot{p}_i + \partial_i V = 0, \tag{8.4}$$

which is Newton's second law of motion with the force identified as the gradient of the potential.

Similarly, in field theory, in discussing a Lagrangian density, $\mathcal{L}$, it is useful to regard the fields ψ^A and their derivatives as independent variables. One then obtains the Euler-Lagrange equation for $\mathcal{L}$:

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi^A)}\right) - \frac{\partial\mathcal{L}}{\partial\psi^A} = 0 \tag{8.5}$$

Both sets of Euler-Lagrange equations given above can be derived using calculus of variations. In electromagnetism, we have the vacuuum Maxwell Lagragian density (5.1), which closely resembles the field theory formulation, with the appropriate replacement of ψ^A by A_α:

$$L = \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta} \tag{8.6}$$

Observe that this Lagrangian density does not contain any source term, hence

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = 0 \tag{8.7}$$

To identify the conjugate momenta π^α to $A_0 = \phi$ and to A_i:

$$\begin{aligned} \pi^{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha} && (8.8) \\ &= \frac{\partial \mathcal{L}}{\partial F_{\gamma\delta}} \frac{\partial F_{\gamma\delta}}{\partial \partial_\beta A_\alpha} && (8.9) \end{aligned}$$

Recall (2.15): $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and since

$$\frac{\partial \partial_\delta A_\gamma}{\partial \partial_\beta A_\alpha} = \delta^\delta_\beta \delta^\gamma_\alpha \tag{8.10}$$

We have:

$$\frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha} = -\frac{1}{4} \left[\eta^{\rho\gamma} \eta^{\sigma\delta} (\delta^\rho_\beta \delta^\sigma_\alpha - \delta^\sigma_\beta \delta^\rho_\alpha) F_{\gamma\delta} + F^{\gamma\delta} (\delta^\gamma_\beta \delta^\delta_\alpha - \delta^\delta_\beta \delta^\gamma_\alpha) \right] \tag{8.11}$$

After simplification, the conjugate momenta are:

$$\begin{aligned} \pi^{\alpha\beta} &= F^{\alpha\beta} && (8.12) \\ \pi^\alpha \equiv \pi^{\alpha 0} &= F^{\alpha 0} = \vec{E} && (8.13) \\ \pi^{\alpha i} &= F^{\alpha i} && (8.14) \end{aligned}$$

Now we compute the Euler-Lagrange equation as follows:

$$\begin{aligned} \partial_\beta \left(\frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha} \right) - \frac{\partial \mathcal{L}}{\partial A_\alpha} &= 0 && (8.15) \\ \partial_\beta F^{\alpha\beta} &= 0 && (8.16) \end{aligned}$$

Recall section 2 to see that (8.16) is actually Maxwell equations without the source of current.

Finally, we will look at the Hamiltonian formulation. In classical mechanics and field theory, the Hamiltonian and Hamiltonian density are, respectively:

$$\begin{aligned} H &= p_i v^i - L(\vec{x}, \vec{v}) && (8.17) \\ \mathcal{H} &= \pi_A \partial_0 \psi^A - \mathcal{L}(\psi^{\mathcal{A}}, \partial_\alpha \psi^\alpha) && (8.18) \end{aligned}$$

Hence, for electromagnetism, using the derived equations above for the conjugate momentum and Eqn. (5.4), the Hamiltonian density can be expressed as the following:

$$\begin{aligned} \mathcal{H} &= \pi^\alpha \partial_0 A_\alpha - \mathcal{L}(\mathcal{A}_\alpha, \partial_\beta \mathcal{A}_\alpha) && (8.19) \\ &= F^{\alpha 0} \partial_0 A_\alpha - \frac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2) && (8.20) \\ &= \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) && (8.21) \end{aligned}$$

Therefore, the Hamiltonian density is simply the energy density, T^{00}.