① $$\partial_\alpha T^{\alpha\beta} = \partial_\alpha\left(-\frac{1}{4\pi}F^{\alpha\gamma}F^{\beta}{}_{\gamma} + \frac{1}{16\pi}g^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}\right)$$

$$= \frac{1}{4\pi}\left(-\partial_\alpha\left(F^{\alpha\gamma}F^{\beta}{}_{\gamma}\right) + \frac{1}{4}g^{\alpha\beta}\partial_\alpha\left(F^{\gamma\delta}F_{\gamma\delta}\right)\right)$$

$$= \frac{1}{4\pi}\left(-\partial^\alpha\left(F_{\alpha\gamma}F^{\beta\gamma}\right) + \frac{1}{2}F_{\gamma\delta}\left(\partial^\beta F^{\gamma\delta}\right)\right)$$

(by expanding the 2nd term)

$$-\partial^\alpha\left(F_{\alpha\gamma}F^{\beta\gamma}\right) = -F^{\beta\gamma}\underbrace{\partial^\alpha F_{\alpha\gamma}}_{\mu_0 J_\gamma} - \underbrace{F_{\alpha\gamma}\partial^\alpha F^{\beta\gamma}}$$

$$\xrightarrow[\text{and } \gamma\to\delta]{\text{rename } \alpha\to\gamma} \quad -F_{\gamma\delta}\partial^\gamma F^{\beta\delta}$$

$$\Rightarrow \partial_\alpha T^{\alpha\beta} = \frac{1}{4\pi}\left(-\mu_0 J_\gamma F^{\beta\gamma} - F_{\gamma\delta}\partial^\gamma F^{\beta\delta} + \frac{1}{2}F_{\gamma\delta}\partial^\beta F^{\gamma\delta}\right) \qquad (*)$$

Index swapping trick:

$$F_{ij}\partial^i F^{mj} = \frac{1}{2}F_{ij}\partial^i F^{mj} + \frac{1}{2}\underbrace{F_{ji}}_{=-F_{ij}}\partial^j F^{mi}$$

(swap $i \leftrightarrow j$ in the 2nd term)

(antisymmetric property of $F_{\mu\nu}$).

$$= \frac{1}{2}F_{ij}\left(\partial^i F^{mj} - \partial^j F^{mi}\right)$$

$$\Rightarrow \frac{1}{4\pi}F_{\gamma\delta}\left(-\partial^\gamma F^{\beta\delta} + \frac{1}{2}\partial^\beta F^{\gamma\delta}\right) = \frac{1}{8\pi}F_{\gamma\delta}\left(\underbrace{-\partial^\gamma F^{\beta\delta}}_{\partial^\gamma F^{\delta\beta}} + \partial^\delta F^{\beta\gamma} + \partial^\beta F^{\gamma\delta}\right)$$

$$= \frac{F_{\gamma\delta}}{8\pi}\left(\partial^\gamma\partial^\delta A^\beta - \partial^\gamma\partial^\beta A^\delta + \partial^\delta\partial^\beta A^\gamma - \partial^\delta\partial^\gamma A^\beta + \partial^\beta\partial^\gamma A^\delta - \partial^\beta\partial^\delta A^\gamma\right).$$

The derivatives commute (i.e., $[\partial^\alpha, \partial^\beta] = 0$)

$$= \partial^\alpha\partial^\beta - \partial^\beta\partial^\alpha$$

$\Rightarrow$ The last 2 terms in the sum in equation (*) cancel each other and we are left with:

$$\partial_\alpha T^{\alpha\beta} = -\frac{\mu_0}{4\pi}J_\gamma F^{\beta\gamma} \qquad \blacksquare$$

② $\beta = 0 \Rightarrow \partial_\alpha T^{\alpha 0} = -\frac{\mu_0}{4\pi} F^{0\gamma} J_\gamma$

For $\gamma = 0$: $F^{00} = 0$ $\Rightarrow$ The right hand side reduces to

$$-\frac{\mu_0}{4\pi} F^{0i} J_i \qquad (i = 1, 2, 3).$$

From the field tensor definition: $F^{0i} = \frac{E_i}{c}$

$$\Rightarrow \partial_\alpha T^{\alpha 0} = -\frac{\mu_0}{4\pi c} E_i J_i = -\frac{\mu_0}{4\pi c} (\vec{E} . \vec{J})$$

For the LHS: $\alpha = 0$: $\partial_0 T^{00} = \frac{1}{c} \partial_t \left(\frac{1}{8\pi} (\vec{E}^2 + \vec{B}^2) \right) = \frac{1}{c} \partial_t \mathcal{E}$

$\alpha = i$: $\partial_i T^{i0} = \partial x_i \left(\frac{1}{4\pi} (\vec{E} \times \vec{B})^i \right) = \partial x_i \frac{\vec{S}^i}{c}$

(can be verified from the definition of the energy-momentum tensor, also see section 8.2.2 in Griffiths' book).

$$\Rightarrow \partial_\alpha T^{\alpha 0} = \frac{1}{c} \left(\partial_t \mathcal{E} + \partial_{x_i} \vec{S}^i \right) = -\frac{\mu_0}{4\pi c} (\vec{E} . \vec{J})$$

$$\rightarrow \frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} . \vec{S} = -\frac{\mu_0}{4\pi} \vec{E} . \vec{J} \qquad \blacksquare$$

③ $\beta = i \Rightarrow \partial_\alpha T^{\alpha i} = -\frac{\mu_0}{4\pi} F^{i\gamma} J_\gamma = -\frac{\mu_0}{4\pi} \big(\underbrace{F^{i0}}_{-E^i/c} \underbrace{J_0}_{c\rho} + \underbrace{F^{ij}}_{\varepsilon^{ijk} B^k} \underbrace{J_j}_{-J^j} \big)$

$$= \frac{\mu_0}{4\pi} \big(E^i \rho + \underbrace{\varepsilon^{ijk} B^k J^j}_{(\vec{J} \times \vec{B})^i} \big) = \frac{\mu_0}{4\pi} (\rho E + \vec{J} \times \vec{B})^i$$

$$\partial_\alpha T^{\alpha i} = \frac{1}{c} \partial_t T^{0i} + \frac{\partial}{\partial x_j} T^{ji} = \frac{1}{c} \partial_t \left(\frac{\vec{S}^i}{c} \right) + \frac{\partial}{\partial x_j} T^{ji}$$

$$\Rightarrow \partial_t \left(\frac{\vec{S}_i}{c^2} \right) + \frac{\partial}{\partial x_j} T^{ji} = \frac{\mu_0}{4\pi} (\rho E + \vec{J} \times \vec{B})^i \qquad \blacksquare$$

Recall $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ in classical E&M.

$\Rightarrow f^i \propto (\rho \vec{E} + \vec{J} \times \vec{B})^i$ is the Lorentz force per unit volume.