PHYSICS 110B – Spring 2011
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Discussion Problems

– Week 2 –

Schutz – Problem 1.8

<u>Invariance of the spacetime interval</u>

Define the interval between any 2 events in as:

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \qquad [1.1]$$

in the coordinate system O.
Assuming a linear relation between the coordinates of O and those of another coordinate system O', and assuming that the origins coincide (i.e., that the events $t' = x' = y' = z' = 0$ and $t = x = y = z = 0$ are the same), we can write the interval in O' as follows:

$$\Delta s'^2 = -\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} M_{\alpha\beta}\left(\Delta x^\alpha \Delta x^\beta\right) \qquad [1.2]$$

for some numbers $\{M_{\alpha\beta}; \alpha, \beta = 0, \dots, 3\}$, which maybe a function of v, the relative velocity between the two frames.
Suppose $\Delta s^2 = 0$, and denote $\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. Then:

$$\Delta s'^2 = 0; \quad \Delta t = \Delta r$$

(a) Summing over all different cases of $\alpha = \beta = 0$, $\alpha = 0$ & $\beta \neq 0$ (and vice versa), $\alpha \neq 0$ & $\beta \neq 0$, we get:

$$\Delta s'^2 = M_{00}\Delta r^2 + 2\left(\sum_{i=1}^{3} M_{0i}\Delta x^i\right)\Delta r + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\left(\Delta x^i \Delta x^j\right) \qquad [1.3]$$

Convention sidenote: In the n+1 dimensional spacetime, Greek letters denote the indices from 0 to n (i.e., both time and space indices), Roman letters denote indices from 1 to n (i.e., only spatial indices).

Note that in the above expression, we have assumed that $M_{0i} = M_{i0}$.

(b) Since $\Delta s'^2 = 0$ for any $\{\Delta x^i\}$, replace Δx^i by – Δx^i in Eq. [1.3] and subtract the resulting equation from [1.3] to establish that $M_{0i} = 0$ for $i = 1, 2, 3$.

$$\Delta s'^2 = M_{00}\Delta r^2 - 2\left(\sum_{i=1}^{3} M_{0i}\Delta x^i\right)\Delta r + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\left(\Delta x^i \Delta x^j\right)$$

$$\rightarrow M_{0i} = -M_{0i} = 0 \ \forall i$$

(c) Use Eq. [1.3] with $\Delta s'^2 = 0$ to establish that:

$$\boldsymbol{M_{ij} = -(M_{00})\delta_{ij}}$$

From [1.3] and the result of part (b):

$$M_{00}\Delta r^2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\left(\Delta x^i \Delta x^j\right)$$

$$M_{00}\Delta r^2 = M_{00}\sum_{i=1}^{3}\sum_{j=1}^{3} \Delta x^i \Delta x^j \delta_{ij}$$

$$\rightarrow M_{00}\delta_{ij} = -M_{ij}$$