PHYSICS 110B – Spring 2011
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Discussion Problems

– Week 3 –

Energy-Momentum Tensor and the Electromagnetic Forces

(Source: http://peeterjoot.wordpress.com/2011/04/10/phy450hs1-relativistic-electrodynamics-problem-set-6/)

Recall important definitions:
Field tensor: (Griffiths 12.118)

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Current density 4-vector: (Griffiths 12.12)

$$J^\mu = \left(c\rho, J_x, J_y, J_x\right) = (c\rho, \mathbf{J})$$

Continuity equation:

$$\nabla . \mathbf{J} = -\partial\rho/\partial t$$

Relativistic potential (Griffiths 12.131) and its relationship with the field tensor:

$$A^\mu = \left(\frac{V}{c}, A_x, A_y, A_z\right)$$

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

Maxwell equations in terms of the relativistic potential, and the current density: (Griffiths 12.136):

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu = -\mu_0 J^\mu$$

where μ_0 is the usual permeability of free space.
The energy-momentum tensor

$$T^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\sigma} F_\sigma^\nu + \frac{1}{16\pi} g^{\mu\nu} F^{\sigma\tau} F_{\sigma\tau}$$

is a conserved quantity:

$$\partial_\mu T^{\mu\nu} = 0$$

1. Conservation relation in the presence of sources:

In the presence of sources, the equation of motion is given by: (Griffiths 12.126)

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$$

Show that:

$$\partial_\alpha T^{\alpha\beta} = -\frac{\mu_0}{4\pi} F^{\beta\gamma} J_\gamma$$

2. Timelike component of the conservation equation:

Consider the $\beta = 0$ components in the above equation. Show that it implies the following energy conservation equation:

$$\frac{\partial \epsilon}{\partial \mathrm{t}} + \nabla . \mathbf{S} = -\frac{\mu_0}{4\pi} \mathbf{E} . \mathbf{J}$$

where ϵ is the energy of the electromagnetic field and **S** is the Poynting vector.

3. Spacelike component of the conservation equation:

Now consider the $\beta = i$ components of the equation in part 1. Show that:

$$\partial_t \left(\frac{\mathbf{S}^i}{c^2} \right) + \partial_{x_j} T^{ji} = \frac{\mu_0}{4\pi} \left(\rho \mathbf{E}^i + (\mathbf{J} \times \mathbf{B})^i \right) \equiv \boldsymbol{f}^i$$

Give a physical interpretation of $\boldsymbol{f}^i$.