PHYS 110B

HW9 - Solution

① (a) $n \sin\theta_c = n_{air} \sin \pi/2 = 1 \Rightarrow \theta_c = \arcsin\left(\frac{1}{n}\right) = 0.675 \text{ rad} = 38.68°$

(b) For total internal reflection: $\theta \leq \frac{\pi}{2} - \theta_c \rightarrow \sin\theta \leq \sin\left(\frac{\pi}{2} - \theta_c\right) = \cos\theta_c$

( $2d\sin\theta = m\lambda_n$ is no longer valid for wave guide with dielectric material.)

For TE mode: $\tan\frac{\varphi_r}{2} = \sqrt{\frac{\cos^2\theta_c}{\sin^2\theta} - 1}$

$$\frac{2\pi}{\lambda} 2d\sin\theta - 2\varphi_r = 2\pi m \Rightarrow \varphi_r = \frac{2\pi}{\lambda} d\sin\theta - \pi m \Rightarrow \frac{\varphi_r}{2} = \frac{\pi}{\lambda} d\sin\theta - m\frac{\pi}{2}$$

$$\Rightarrow \tan\left(\frac{\pi d}{\lambda}\sin\theta - m\frac{\pi}{2}\right) = \sqrt{\frac{\cos^2\theta_c}{\sin^2\theta} - 1} \quad (*) \quad \left(\lambda = \frac{\lambda_0}{n}\right)$$

$m=0 \Rightarrow \tan\left(\frac{\pi n d}{\lambda_0}\sin\theta\right) = \sqrt{\frac{\cos^2\theta_c}{\sin^2\theta} - 1} \rightarrow \theta \approx 6.51°$

$\rightarrow \boxed{\theta_b = 90° - 6.51° = 83.49°}$

By graphing the two sides of equation (*), we can obtain the number of modes

$\rightarrow$ number of modes $= 7$.

(c) $v_g = \frac{d\omega}{dk_z}$, $k_z = \sqrt{(nk_0)^2 - k_\perp^2} = \sqrt{\left(n\frac{\omega}{c}\right)^2 - k_\perp^2}$

for $m=0$ : $k_\perp d = -\varphi_r$

$$\Rightarrow v_g^{-1} = \frac{dk_z}{d\omega} = \frac{d}{d\omega}\left(\left[\left(n\frac{\omega}{c}\right)^2 - \frac{\varphi_r^2}{d^2}\right]^{1/2}\right)$$

$\Rightarrow v_g = 0.62c$.

② (a) $n_1 = 1.6, \quad n_2 = 1.4.$

$n_1 \sin\theta_c = n_2 \sin 90^\circ \Rightarrow \theta_c = \arcsin\left(\frac{n_2}{n_1}\right) = 61.04^\circ \approx 1.06 \text{ rad}.$

(b) Use a similar approach to #(1b).

$\Rightarrow$ Number of TE modes $= 4$. $(m = 0, 1, 2, 3)$.

③ (a) $\nu_f = \frac{v_p}{2L}$ ($v_p$: effective phase velocity in the cavity, $L$: cavity length).

$$\Rightarrow \nu_f = \frac{c}{2(15\times10^{-2}\text{ m})} = 1 \text{ GHz}.$$

(b) A plate of thickness 2.5 cm & $n = 1.5$ in this 15 cm cavity is optically equivalent to a $(15 - 2.5) + 2.5 \times 1.5 \doteq 16.25$ cm cavity of air

$$\Rightarrow \nu_f' = \frac{c}{2(16.25\times10^{-2}\text{ m})} = 0.92 \text{ GHz}.$$

④
$$E_t = E_0 t_1 t_2 e^{i\delta} + E_0 t_1 t_2 r_1 r_2 e^{3i\delta} + E_0 t_1 t_2 r_1^2 r_2^2 e^{5i\delta} + \cdots$$
$$= E_0 t_1 t_2 e^{i\delta}\left(1 + r_1 r_2 e^{2i\delta} + (r_1 r_2 e^{2i\delta})^2 + \cdots\right)$$
$$= \frac{E_0 t_1 t_2 e^{i\delta}}{1 - r_1 r_2 e^{2i\delta}}$$

$$\frac{I_t}{I_0} = \left|\frac{E_t}{E_0}\right|^2 = \left|\frac{t_1 t_2}{1 - r_1 r_2 e^{2i\delta}}\right|^2 = \frac{T_1 T_2}{1 + R_1 R_2 - 2 r_1 r_2 \cos 2\delta}$$

$$T = 1 - R, \quad \delta = k_0 \Delta L = \frac{2\pi\nu}{c} d.$$

$$\Rightarrow \frac{I_t}{I_0} = \frac{(1-R_1)(1-R_2)}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\cos\left(\frac{4\pi}{c} d\,\nu\right)}$$

$$R_1 = 0.98, \quad R_2 = 0.99, \quad d = 0.001 \text{ m}$$

$$\Rightarrow \frac{I_t}{I_0} = \frac{2\times10^{-4}}{1.9702 - 1.96997\cos(4.19\times10^{-11}\,\nu)}$$