SOLUTIONS for Physics 110 B Midterm Spring 2012

1. [25] The ladder in the barn. Alice carries a ladder of proper length L_p at a velocity $v = (4/5)c$ into a barn, entering at the left. In Bob's CS, the ladder is fully inside the barn at time $t = 0$, with the left end of the ladder coincident with the left-hand door of the barn. Alice and Bob synchonize their clocks so that at the left barn door she measures $t' = 0$ when Bob measures $t = 0$.

a) [10] What time does Alice measure at the right hand end of the ladder when Bob determines that the entire ladder has just entered the barn (i.e., at $t = 0$ in his frame)? Label this event "A"; the question is, what is t'_A? Your answer should be in terms of L_p and c.

Solution: $\beta = 4/5 \Rightarrow \gamma = 5/3$. At $t = 0$, the first instant at which the ladder is fully in the barn, $x = L_p/\gamma$, where $x = 0$ corresponds to the entrance to the barn. This value of x is x_A, and $t = t_A$. Then

$$t'_A = \gamma(t_A - \beta x_A/c) = -\gamma\beta x_A/c = -\gamma\beta L_p/\gamma c = -\frac{4L_p}{5c} \tag{1}$$

b) [15] Let event B be the location of the left edge of the ladder when Alice measures a time t'_A on her clocks. What is the location of the left edge of the ladder in Alice's frame at that time—i.e., what is x'_B? What is it in Bob's frame—i.e., what is x_B? Again, express your answer in terms of L_p and c.

Solution: First, note that

$$x'_A = \gamma(x_A - c\beta t_A) = \gamma x_A = \gamma(L_p/\gamma) = L_p. \tag{2}$$

This makes sense since the right-hand edge of the ladder is always at $x' = L_p$. Correspondingly, the left-hand edge,

$$x'_B = x'_A - L_p = 0. \tag{3}$$

The location of this point in Bob's frame is

$$x_B = \gamma(x'_B + c\beta t'_A) = \gamma\left[0 + c\beta\left(-\frac{4L_p}{5c}\right)\right] = -\frac{16}{15}\,L_p \tag{4}$$

2. [20] Relativistic beaming. Alice travels in a rocket that moves at a velocity corresponding to a Lorentz factor γ. She shines a flashlight at 90 degrees to the direction in which the rocket is traveling. Bob, who is stationary, measures an angle θ between the flashlight beam and the direction of motion of the rocket; what is $\tan\theta$? This effect is called relativistic beaming, and is important in making gamma-ray bursts visible to the edge of observable universe.

Solution: Let the velocity of Allce's rocket be v, and denote her frame with a prime. $\tilde{k}$ is a 4-vector, so

$$k'_{\|} = \gamma\left(k_{\|} - \frac{v\omega}{c^2}\right) = \gamma(k_{\|} - \beta k), \tag{5}$$

since $\omega = ck$. Since Alice shines the light at 90 degrees, $k'_{\|} = 0$, which implies

$$\cos\theta = \frac{k_{\|}}{k} = \beta, \tag{6}$$

so that

$$\tan\theta = \frac{(1-\cos\theta^2)^{1/2}}{\cos\theta} = \frac{1}{\gamma\beta} = \frac{1}{(\gamma^2-1)^{1/2}}. \tag{7}$$

(Note that for large γ, this implies $\tan\theta \to 1/\gamma$, so the light is very nearly parallel in Bob's frame.)

3. [20] The scattering of a photon by an initially static electron is often analyzed in the center-of-momentum frame, in which the total 3-momentum is zero. A photon of energy $2m_ec^2$ moving to the right strikes an electron at rest. What is the velocity of the center of momentum frame?

Solution: Let p be the photon momentum and p_e the electron momentum. Since $E = pc$, the initial photon momentum is $p = 2m_ec$. Denote the center of momentum frame with a prime. By definition, we have

$$p' + p'_e = 0 \tag{8}$$

For the photon, we have

$$p' = \gamma_r\left(p - \beta_r\frac{E}{c}\right) = \gamma_r 2m_ec(1-\beta_r), \tag{9}$$

whereas for the electron, which has $E = m_ec^2$, we have

$$p'_e = \gamma_r(0 - \beta_r m_ec^2) = -\gamma_r\beta_r m_ec^2. \tag{10}$$

Equation (8) then implies

$$\gamma_r\beta_r = 2\gamma_r(1-\beta_r) \quad\Rightarrow\quad \frac{1}{\beta_r} - 1 = \frac{1}{2} \quad\Rightarrow\quad \beta_r = \frac{2}{3}. \tag{11}$$

4. [35] (Problem 11.14): The Bohr model of the atom has an electron orbiting a proton at a radius r_0. The charge of the electron is $-e$, and its mass is m_e. If classical physics applied to atoms, the electron would radiate and spiral into the proton. Assuming that the electron always moves in a circular orbit and that its velocity is always much less than c, determine how long it would take for a classical electron to spiral into the proton.

Solution: Let E be the energy of the electron. It is the sum of the kinetic and potential energies:

$$E = \frac{1}{2}m_ev^2 - \frac{e^2}{4\pi\epsilon_0 r}. \tag{12}$$

Force balance

$$\frac{m_ev^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} \tag{13}$$

then implies

$$E = -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0 r}\right). \tag{14}$$

The rate at which the electron loses energy is given by the Larmor radiation formula, since we are assuming that the motion is non-relativistic:

$$\frac{dE}{dt} = -\frac{\mu_0 e^2 a^2}{6\pi c} \tag{15}$$

Now the acceleration is

$$a = \frac{v^2}{r} = \frac{e^2}{4\pi\epsilon_0 m_e r^2}, \tag{16}$$

from equation (13). With the aid of equation (14), we find

$$\frac{d}{dt}\left(-\frac{e^2}{8\pi\epsilon_0 r}\right) = -\frac{\mu_0 e^2}{6\pi c}\left(\frac{e^2}{4\pi\epsilon_0 m_e r^2}\right)^2, \tag{17}$$

$$\frac{e^2}{8\pi\epsilon_0 r^2}\frac{dr}{dt} = -\frac{\mu_0 e^6}{6\pi c \cdot 16\pi^2\epsilon_0^2 m_e^2 r^4}, \tag{18}$$

$$\frac{dr}{dt} = -\frac{\mu_0 e^4}{12\pi^2 c\epsilon_0 m_e^2 r^2}. \tag{19}$$

Integration gives

$$\frac{1}{3}r^3 = -\frac{\mu_0 e^4}{12\pi^2 c\epsilon_0 m_e^2} \cdot t + C, \tag{20}$$

where C is the integration constant. Since $r = r_0$ at $t = 0$, we have $C = r_0^3/3$. Hence the electron reaches the proton at $r = 0$ at a time

$$t = \frac{4\pi^2 c\epsilon_0 m_e^2 r_0^3}{\mu_0 e^4}. \tag{21}$$