4. A "fan" of N plane waves are propagating symmetrically with respect to the z axis, as shown in figure D below. The angular spacing between successive members of the fan is fixed and equal to $\Delta\theta$. Describe the interference pattern observed on a plane perpendicular to the z axis.

Interference Problems

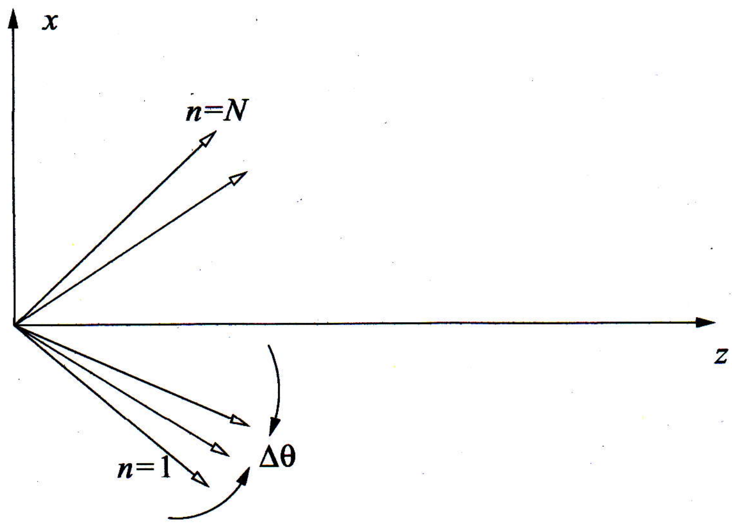
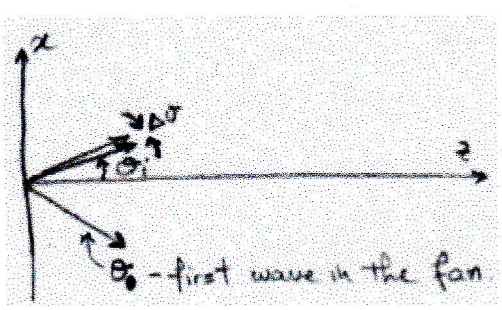


Figure D

5. Describe the interference pattern between two counter-propagating plane waves. This is also known as a "standing wave." Explain why.

Solutions

4. We sketch the system as follows:



The m^{th} plane wave is at an angle $\theta_m = \theta_0 + m\Delta\theta$

$$E_m = e^{i\frac{2\pi}{\lambda}[\cos\theta_m z + \sin\theta_m x]}$$

Assuming small angles (paraxial approximation), $\theta_m \ll 1$

$$\cos\theta_m \approx 1, \sin\theta_m \approx \theta_m = \theta_0 + m\Delta\theta$$
$$E_m \approx e^{i\frac{2\pi}{\lambda}(z+\theta_m x)} = e^{i\frac{2\pi}{\lambda}(z+\theta_0 x + m\Delta\theta x)}$$

Adding all the plane waves,

$$
\begin{aligned}
E_T &= \sum_{m=0}^{N-1} E_m \\
&= \sum_{m=0}^{N-1} e^{i\frac{2\pi}{\lambda}(z+\theta_0 x+m\Delta\theta x)} \\
&= e^{i\frac{2\pi}{\lambda}z} e^{i\frac{2\pi}{\lambda}\theta_0 x} \underbrace{\sum_{m=0}^{N-1} (e^{i\frac{2\pi}{\lambda}x\Delta\theta})^m}_{\text{Geometric series: } \theta_0=1,\ r=e^{i\frac{2\pi}{\lambda}x\Delta\theta}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore E_T &= e^{i\frac{2\pi}{\lambda}z} e^{i\frac{2\pi}{\lambda}\theta_0 x} \cdot \frac{1-e^{i(\frac{2\pi}{\lambda}Nx\Delta\theta)}}{1-e^{i(\frac{2\pi}{\lambda}x\Delta\theta)}} = e^{i\frac{2\pi}{\lambda}z} e^{i\frac{2\pi}{\lambda}\theta_0 x} \cdot \frac{1-e^{i\phi_1}}{1-e^{i\phi_2}} \\
&= e^{i\frac{2\pi}{\lambda}z} e^{i\frac{2\pi}{\lambda}\theta_0 x} \cdot \frac{e^{i\frac{\phi_1}{2}}}{e^{i\frac{\phi_2}{2}}} \cdot \frac{e^{-i\frac{\phi_1}{2}} - e^{i\frac{\phi_1}{2}}}{e^{-i\frac{\phi_2}{2}} - e^{i\frac{\phi_2}{2}}} \\
|E_T|^2 &= \left| \frac{e^{-i\frac{\phi_1}{2}} - e^{i\frac{\phi_1}{2}}}{e^{-i\frac{\phi_2}{2}} - e^{i\frac{\phi_2}{2}}} \right|^2 = \left(\frac{2i\sin(\frac{\phi_1}{2})}{2i\sin(\frac{\phi_2}{2})} \right)^2 = \frac{\sin^2(\frac{\phi_1}{2})}{\sin^2(\frac{\phi_2}{2})}
\end{aligned}
$$

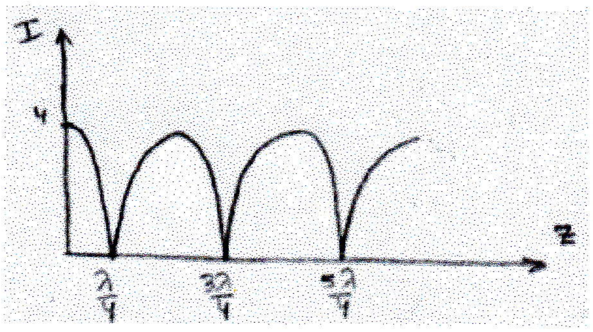
5. Forward propagating plane wave: $\vec{k}_1 = \frac{2\pi}{\lambda}\hat{z}$

   Backward propagating plane wave: $\vec{k}_2 = -\frac{2\pi}{\lambda}\hat{z}$

$$
E_1 = e^{i(\frac{2\pi}{\lambda}z-\omega t)}, \quad E_2 = e^{i(-\frac{2\pi}{\lambda}z-\omega t)}
$$

$$
I = 2(1+\cos\Delta\phi), \text{ where } \Delta\phi = \phi_1 - \phi_2 = \frac{4\pi}{\lambda}z
$$

$$
\therefore I = 2\left[1+\cos\left(\frac{4\pi}{\lambda}z\right)\right] = 4\cos^2\left(\frac{2\pi}{\lambda}z\right)
$$



Note that although we did not ignore the time dependence of each wave (ωt), the interference wave is independent of time, thus the term "standing wave."

Recall:

Fabry – Perot Interferometer:

$\delta = \frac{2\pi}{\lambda} 2nl\cos\theta$: phase diff. between each succeeding reflection

If both surfaces have reflectance R, transmittance function of the interferometer is:

$$T = \frac{(1-R)^2}{1+R^2-2R\cos\delta} = \frac{1}{1+F\sin^2(\delta/2)}$$

$F = \frac{4R}{(1-R)^2}$: coeff. of finesse

Max transmission : $T=1 \Leftrightarrow$ optical path length diff. ($2nl\cos\theta$) b.w transmitted beam is $m\lambda$.

$$T+R=1 \Rightarrow R_{max} = 1-\frac{1}{1+F} = \frac{4R}{(1+R)^2} \quad \text{when } 2nl\cos\theta = (m+\tfrac{1}{2})\lambda.$$

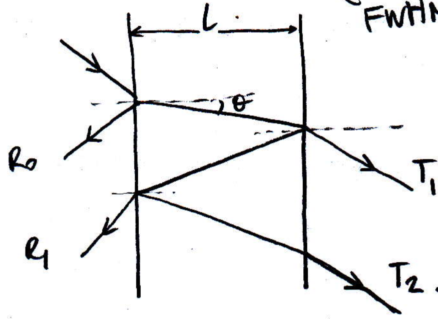
Wavelength separation b.w adjacent peaks : "free spectral range" $\Delta\lambda$: (FSR)

$$\Delta\lambda = \frac{\lambda_0^2}{2nl\cos\theta+\lambda_0} \approx \frac{\lambda_0^2}{2nl\cos\theta}$$

(λ_0: central wavelength of nearest transmission peak)

$$(\text{Finesse}) \rightarrow \mathcal{F} = \frac{\Delta\lambda}{\delta\lambda} = \frac{\pi}{2\sin^{-1}(1/\sqrt{F})} \approx \frac{\pi\sqrt{F}}{2} = \frac{\pi\sqrt{R}}{1-R}$$

($\delta\lambda$ ↖ FWHM.)



5 Fabry-Perot Cavities I

A particular semiconductor laser oscillates at a nominal wavelength of $\lambda_0 = 8800$ Å, but its spectrum really consists of two closely-spaced modes $\lambda_1 = 8800.WXYZ$ Å and $\lambda_2 = 8800.IJKL$ Å. To resolve the wavelength difference between the modes, we use a scanning Fabry-Perot cavity in which the

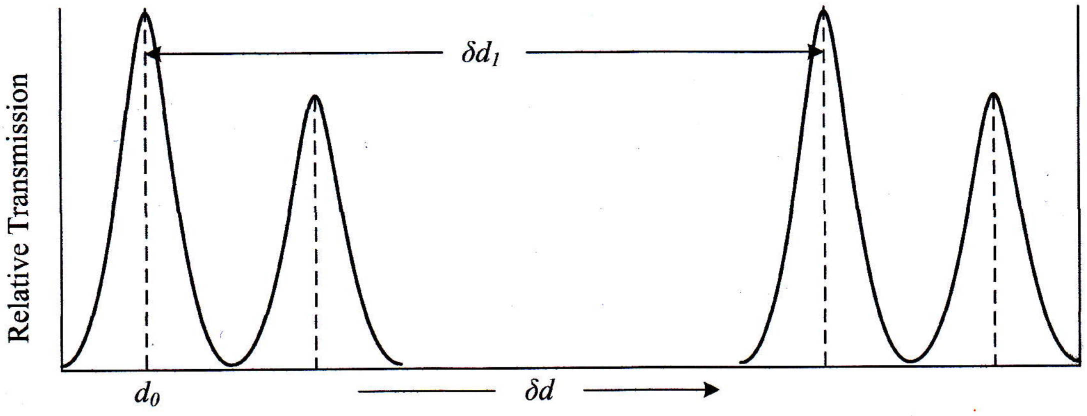


Figure 4: The transmission of a two longitudinal mode semiconductor laser through a Fabry-Perot cavity as d is increased

mirror spacing is changed from $d_0 = 4$ cm to $d_0 + \delta d$ with $\delta d \ll d_0$. The relative transmission through the cavity is shown (to scale) in Figure 4 as a function of the change δd. You may need to use a ruler to measure relative distances for this problem. The solution is non-unique. You may find any valid solution.

(a) What is the distance δd_1 shown in Figure 4

(b) The major peaks are the resonances associated with λ_1 and the minor ones are associated with λ_2. Estimate $\lambda_2 - \lambda_1$.

(c) What is the finesse, F, of the Fabry-Perot cavity?

Solution

1. δd_1 is simply one half-wavelength of light. In this case:

$$\delta d_1 = \frac{\lambda_0}{2} = \frac{8800 \text{ Å}}{2} = 440 \text{ nm}$$

2. First, we realize that the two peaks must come from the same longitudinal mode, q, and that the mode number q is related to the distance between the mirrors via:

$$q = \frac{2nd}{\lambda_0}$$

We already know that $\delta d_1 = 440$ nm corresponds to a FSR, and we estimate that the distance between the primary and secondary peaks, then corresponds to:

$$\delta d_2 = \frac{563}{2252}(440 \text{ nm}) = 110.0 \text{ nm}$$

By equating mode numbers, we have:

$$\frac{2d_0}{\lambda_1} = \frac{2\left(d_0 + \delta d_2\right)}{\lambda_2}$$

$$d_0\lambda_2 - d_0\lambda_1 = \delta d_2\lambda_1$$

$$\lambda_2 - \lambda_1 = \frac{\delta d_2\lambda_1}{d_0} = \frac{(110.0\ \text{nm})\left(8800\ \text{Å}\right)}{(4\ \text{cm})} = 2.42\ \text{pm}$$

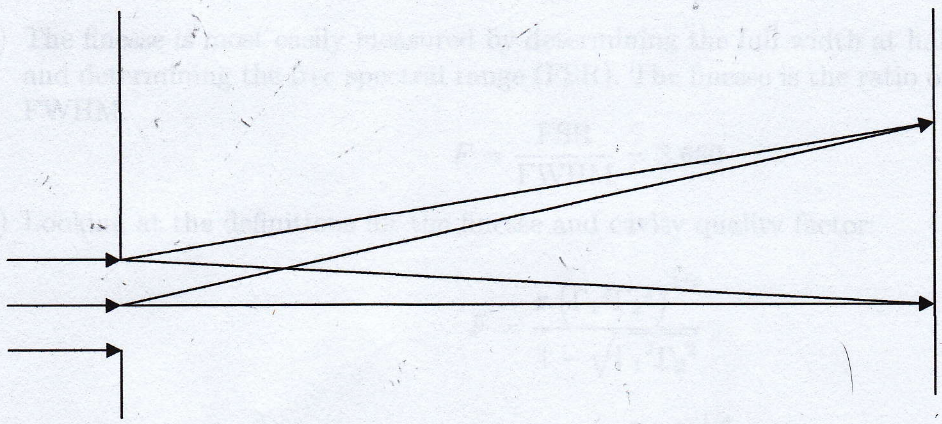
3.

$$F = \frac{\text{FSR}}{\text{FWHM}} = \frac{2252}{199} = 11.32$$

Diffraction

5) Calculate the separation between the main image and the first-order diffraction peak for a 1.0 angstrom X-ray imaged 1.0 m away from a grating with 1.0 micron spacing. Do you think conventional diffraction gratings work for X-rays? What is used instead?

In a diffraction grating, I have:



A diffraction peak occurs where $d_{edge} - d_{center} = n\lambda$. Then, assuming that by "main image" the question means the center of the diffraction pattern, this distance will be given by: $d_{edge} - d_{center} = \pm 1\lambda = \pm\lambda$

$$\sqrt{\left(\frac{10^{-6}}{2}m - d_{peak}\right)^2 + (1.0m)^2} - \sqrt{d_{peak}^2 + (1.0m)^2} = \pm 10^{-10}m$$

Solving numerically with Mathematica, I have:

$$d_{peak} \approx 0.2mm$$

This is identical to the result (in the $\sin\theta \approx \theta \quad \theta \approx 0$ approximation) given by

$$a\sin\theta = m\lambda$$

$$a = \frac{10^{-6}}{2} meters$$

$$m = 1$$

$$\lambda = 10^{-10} m$$

$$D = 1m$$

$$D\frac{\lambda}{a} = 0.2mm$$

This is very small. The probably works a lot better with a finer-spaced grating, closer to the x-ray wavelength so as to space out the peaks, perhaps a crystal or thin sheet of foil.

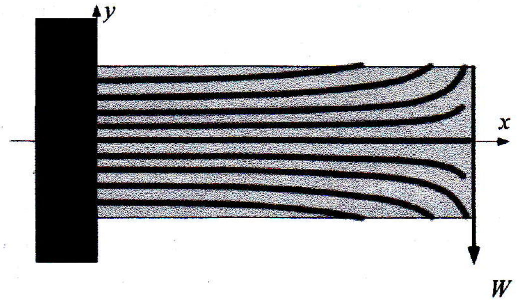


Fig. vi.5. Photoelastic fringes in a stressed cantilever.

Recall:
HWP: $\Delta\varphi = \pi$ (phase shift)
$\Delta\varphi = k_0 d(n_1 - n_2)$
↑
thickness.

6.17 A right-handed circularly polarized wave with wavelength λ is incident in the z-direction on a half-wave plate made from a crystal with principal refractive indices (for x and y-polarizations) n_1 and n_2.

1. What is the thickness of the plate?
2. Show that the wave exits the plate with left-handed circular polarization
3. Use the fact that the torque exerted by an electric field $\mathbf{E}$ on a dipole $\mathbf{p}$ is $\mathbf{p} \times \mathbf{E}$ to find the torque exerted on the plate as it reverses the circular polarization.
4. Show that your result agrees with the quantum interpretation that right and left-handedly polarized photons have angular momenta of $\pm\hbar$ respectively.

The wave with x-polarization is $E_x = E_0 \cos(k_0 n_1 z)$ and its polarization density is $P_x = \epsilon_0(\epsilon_1 - 1)E_0 \cos(k_0 n_1 z)$. Likewize, the wave with y-polarization is $E_y = \pm E_0 \sin(k_0 n_2 z)$ and its polarization density is $P_y = \pm\epsilon_0(\epsilon_2 - 1)E_0 \sin(k_0 n_1 z)$. The torque per unit area for the positive sign is

$$\tau = \int_0^z \mathbf{E} \times \mathbf{P} \mathrm{d}z; \tag{vi.13}$$

$$\begin{aligned}
\tau_z &= \int_0^z (E_x P_y - E_y P_x) \mathrm{d}z \\
&= \epsilon_0 E_0^2 \int_0^z [(\epsilon_2 - 1)\cos(k_0 n_1 z)\sin(k_0 n_2 z) - (\epsilon_1 - 1)\sin(k_0 n_2 z)\cos(k_0 n_1 z)]\mathrm{d}z \\
&= \tfrac{1}{2}\epsilon_0 E_0^2(\epsilon_2 - \epsilon_1) \int_0^z [\sin[k_0 z(n_1 + n_2)] + \sin[k_0 z(n_1 - n_2)]\mathrm{d}z. \qquad \text{(vi.14)}
\end{aligned}$$

Within a plate of thickness d for which $\cos[k_0 d(n_1 - n_2)] = -1$, which is a half-wave plate and therefore reverses the sense of rotation of a circularly-polarized wave, the first term in the integrand oscillates many times and its integral is negligible, and then we get for the second term, using $\epsilon = n^2$,

$$\begin{aligned}\tau_z &= \tfrac{1}{2}\epsilon_0 \frac{\epsilon_2 - \epsilon_1}{k_0(n_1 - n_2)}[-2] \\ &= \epsilon_0 \frac{E_0^2(n_1 + n_2)}{k_0} = \epsilon_0 \frac{E_0^2 c(n_1 + n_2)}{\omega} \qquad \text{(vi.15)} \\ &= 2W\bar{n}/\omega. \qquad \text{(vi.16)}\end{aligned}$$

where $W = \epsilon_0 E_0^2 c$ is the beam power. If the beam delivers N photons per second, $W = N/\hbar\omega$ and the torque per second per photon, which is the change in angular momentum per photon, is $2\bar{n}\hbar$. Thus a circularly-polarized photon in free space has angular momentum $\pm\hbar$, the sign depending on the sense of the polarization.