PHYS 110B - Practice Final Solutions

① a) Similar to standard double slit problem:

$$U(\psi) \propto 1 + e^{i\beta} + e^{-i\beta} \qquad (\beta = kb\sin\psi)$$

$$U(\psi) \propto 1 + 2\cos\beta$$

$$\Rightarrow \frac{I(\psi)}{I(0)} = \frac{(1+2\cos\beta)^2}{9}$$

$$\text{or } \frac{I(\psi)}{I(0)} = \frac{1}{9}\frac{\sin^2(3\gamma)}{\sin^2\gamma} \qquad (\gamma = \beta/2) \text{ as usual.}$$

b) This configuration is equivalent to a triple-superposition of the triple-slit problem in (a)
→ convolution of the arrangement in (a) with itself.

Fraunhofer conditions: the image is a Fourier transform of the aperture function
Fourier transform of a convolution is the product of Fourier transforms

$$\rightarrow \frac{I(\psi)}{I(0)} = \frac{(1+2\cos\beta)^4}{81}.$$

② $E(z,t) = \mathrm{Re}\left(\tilde{E}\, e^{i(kz-\omega t)}\right)$

$\tilde{E} = E_0\left((2-i)\hat{x} + (1-2i)\hat{y}\right)$

Jones vector: $J = \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix}$

LP with TA along $\hat{x}$: $M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Rotate the LP by φ wrt $\hat{x}$: $R = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$

$$M(\varphi) = R^{-1} M(0) R = \begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}$$

$I_0 \propto J^\dagger J = \begin{pmatrix} 2+i & 1+2i \end{pmatrix}\begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} = 10.$ (before LP).

$I_1 \propto (MJ)^\dagger (MJ) = J^\dagger M^\dagger M J = J^\dagger M J$ (after LP) $(M^\dagger = M,\ M^2 = M)$.

$$= \begin{pmatrix} 2+i & 1+2i \end{pmatrix}\begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}\begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} = 5 + 8\sin\varphi\cos\varphi \leq 9$$

$\hookrightarrow \boxed{\varphi = \pi/4.}$

③ $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$\partial^\mu = \left(\frac{1}{c}\partial_t, -\nabla\right)$, $\partial_\mu = \left(\frac{1}{c}\partial_t, \nabla\right)$

$A^\mu = \left(\frac{V}{c}, \vec{A}\right)$

$J^\mu = (c\rho, \vec{J})$.

④ Thickness of the silver coating should be just above the skin depth b.c the fields do not penetrate much beyond this point.

For silver: $\rho = 1.59 \times 10^{-8}\ \Omega m$ (Griffiths Table 7.1 - resistivity).

$\varepsilon \approx \varepsilon_0$

$\Rightarrow \omega\varepsilon \approx 2\pi \times 10^{10} \times 8.85 \times 10^{-12} = 0.56$

$\sigma = 1/\rho \gg \omega\varepsilon$

$\Rightarrow$ skindepth $d = \frac{1}{\kappa} \approx \sqrt{\frac{2}{\omega\sigma\mu}} = 6.4 \times 10^{-7} m$

(Griffiths Eq. 9.128) $\mu = \mu_0$

$\Rightarrow$ coating should be ~ 0.001 mm.

⑤ Waveguide $a = 2.28 cm$, $b = 1.0 cm$, $\omega = 2\pi f = 2\pi(1.70 \times 10^{10}) \approx 107$ GHz.

Griffiths 9.189: $\omega_{10} = \frac{c\pi}{a} = 4.13 \times 10^{10}$ Hz.

$\omega_{20} = 2\omega_{10} = 8.26 \times 10^{10}$ Hz

$\omega_{30} = 3\omega_{10} = 12.42 \times 10^{10}$ Hz $> \omega$: not allowed.

$\omega_{01} = \frac{c\pi}{b} = 9.42 \times 10^{10}$ Hz

$\omega_{02} = 2\omega_{01} = 18.84 \times 10^{10}$ Hz $> \omega$: not allowed.

$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 102$ GHz.

$\Rightarrow$ Allowed modes: 10, 20, 01, 11.

To get only 1 mode: $\omega_{10} < \omega < \omega_{20} \Rightarrow \frac{c\pi}{a} < \omega < \frac{2c\pi}{a}$

$\omega = \frac{2\pi c}{\lambda}$

$\Rightarrow a < \lambda < 2a$.

Lorentz tranoformation of E & B :

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & -E_y/c & -E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$ (Griffiths 12.118)

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\mu\nu} = \Lambda^{\mu}_{\ \lambda} \Lambda^{\nu}_{\ \sigma} F^{\lambda\sigma}$$ (Griffiths 12.114).

⑥ Griffiths 11.25:

$$p(t) = 2qz(t)$$

$$\ddot{p} = 2q\ddot{z}$$

$$F = m\ddot{z} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2z)^2}$$

$$P = \frac{\mu_0 \ddot{p}}{6\pi c}$$

⑧ $(n-1) k_0 d = 2\pi N$

or use: $N_{glass} = \frac{2d}{\lambda_n}$, $N_{ar} = \frac{2d}{\lambda_0}$, $\Delta N = N_g - N_a = 18.5$.

⑨ Resolving power: $R = \frac{\lambda}{\Delta\lambda} = \frac{Na(\sin\theta_m - \sin\theta_i)}{\lambda} = \frac{\text{width}\,(\sin\theta_m - \sin\theta_i)}{\lambda}$

Assume normal incidence:

$\Rightarrow R = \frac{\lambda}{\Delta\lambda} \simeq \frac{w}{\lambda} \Rightarrow$ width $w = \frac{\lambda^2}{\Delta\lambda}$.

$\Delta\lambda = \frac{\lambda_0^2}{c}\Delta\nu$.

Better resolving power. Fabry-Perot cavity.

⑩ (i) $\alpha_r = \alpha_s + \frac{1}{2d}\ln\left(\frac{1}{R_1 R_2}\right) = \frac{1}{2d}\ln\left(\frac{1}{R^2}\right) = -\frac{1}{d}\ln R$.

$\mathcal{F} = \frac{\pi e^{-\alpha_r d/2}}{1-e^{-\alpha_r d}} = \frac{\pi e^{(\ln R)/2}}{1-R} = 31414$.

(ii) $I_{\text{inside cavity}} = \frac{I_0}{T} = \frac{I_0}{1-R} = 10^4 I_0$.

← incident

$I_{total} = I_{left} + I_{right} = 10^4 I_0 + 10^4 I_0 = 2\times10^4 I_0$.

Fraction of light absorbed by molecule:

$10^{-12} \times 100 \times I_{total} = 10^{-12} \times 100 \times 2\times10^4 = 2\times10^{-6}$.

↑ increase by making area smaller $\left(r^2/r_0^2\right)$

(iii) $I_{\text{inside cavity}} = \frac{T I_0}{(1-T)^2} = \frac{T I_0}{(1-\sqrt{R_1 R_2})^2}$

$(I = R_1 R_2 I_0 \; ; \; E = T E_0 = \sqrt{R_1 R_2}\, E_0$

⑪ Loss = gain

$\Rightarrow (1-R) = T = 0.1(P_{sat} - P) \Rightarrow P = P_{sat} - \frac{T}{0.1}$

↙ actual power inside cavity

$P_{output} = T \times P = T\left(P_{sat} - \frac{T}{0.1}\right)$